UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2022 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The spacetime outside a spherical planet of mass M is described by the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad f(r) = 1 - \frac{2M}{r}$$

where (θ, ϕ) can be regarded as spherical polar coordinates.

A gyroscope orbits the planet. The orbit is circular, with $r \equiv R > 3M$, $\theta \equiv \pi/2$, and angular velocity $d\phi/dt = \Omega$. The spin axis of the gyroscope points in the direction of a vector s^a that undergoes parallel transport around the orbit. When $t = 0$, s^a points in the direction of $\partial/\partial r$.

(a)(i) Show that $s^a u_a$ is constant where u^a is the 4-velocity of the gyroscope. (ii) Deduce that $s^a u_a = 0$.

(b) Show that, in the coordinate basis defined by the Schwarzschild coordinates, the components s^μ satisfy the differential equation

$$\frac{ds^\mu}{dt} + \Gamma^\mu_{t\nu} s^\nu + \Omega \Gamma^\mu_{\phi\nu} s^\nu = 0$$

(c) Show that s^θ vanishes for $t > 0$.

(d) Use the results of part (a) to write s^t in terms of s^ϕ and hence show that s^r and s^ϕ satisfy a pair of first order coupled differential equations. Show that the solution of these equations is

$$s^r = A\cos(\Omega' t) \qquad s^\phi = -\frac{A\Omega}{\Omega' R}\sin(\Omega' t)$$

where A is an arbitrary positive constant and Ω' is a constant that you should determine in terms of M, R, Ω.

(e) Explain why $s^a s_a$ must be constant. Hence determine Ω in terms of M and R. Show that after one orbit the spin axis of the gyroscope has rotated through an angle

$$\alpha = 2\pi \left[1 - \left(1 - \frac{3M}{R}\right)^{1/2}\right]$$

2

(a)(i) The action for a 3-form field H_{abc} is

$$S_H = \int d^4x \sqrt{-g} \left(-\frac{1}{6} H_{abc} H^{abc} \right).$$

What is the energy-momentum tensor of this field? Simplify your answer as much as possible. *[You may assume standard results for variations of the metric.]*

(ii) Consider a theory of gravity in which the "matter" is a covector field ω_a described by a diffeomorphism-invariant action S_{matter} which depends on g_{ab}, ω_a and their derivatives. Let T_{ab} be the energy-momentum tensor and define

$$E^a = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta \omega_a}$$

Show that

$$\nabla_a T^a{}_b = E^a \nabla_b \omega_a - \nabla_a \left(E^a \omega_b \right)$$

and hence deduce that the energy-momentum tensor is conserved if the covector field satisfies its equation of motion. *[You may assume standard results for the Lie derivative.]*

(b) Let Σ be a timelike hypersurface in a Lorentzian spacetime (M, g). Let n_a be a covector field defined in a neighbourhood of Σ such that n_a is normal to Σ and satisfies $n_a n^a = 1$. The extrinsic curvature tensor is defined by $K(X,Y) = -n_a \left(\nabla_{X_\parallel} Y_\parallel \right)^a$ where X, Y are arbitrary vector fields and $\parallel$ denotes projection onto Σ, e.g. $(X_\parallel)^a = h^a_b X^b$ where $h^a_b = \delta^a_b - n^a n_b$.

(i) Explain why this defines a $(0,2)$ tensor and derive the formula $K_{ab} = h^c_a h^d_b \nabla_c n_d$.

(ii) Prove that $K_{ab} = K_{ba}$.

(iii) Let U^a be tangent to an affinely parameterized geodesic in (M, g) that intersects Σ at a point p. Show that the following equation holds at p

$$U^a \nabla_a (U^b n_b) = K_{ab} U^a U^b + f U^b n_b$$

where $f = U^a n^c \nabla_c n_a$.

(iv) Σ is *totally geodesic* iff any geodesic in (M, g) that starts on Σ and is initially tangent to Σ remains within Σ. Prove that if Σ is totally geodesic then $K_{ab} = 0$.

3

(a) The retarded solution of the linearized Einstein equation is

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4\int d^3\mathbf{x}'\,\frac{T_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

(i) Show that

$$\bar{h}_{ij}(t,\mathbf{x}) \approx \frac{2}{r}\ddot{I}_{ij}(t-r)$$

where $i,j\in\{1,2,3\}$, $r=|\mathbf{x}|$, I_{ij} should be defined, and you should explain carefully any assumptions or approximations that you make.

(ii) Show that

$$\bar{h}_{0i}(t,\mathbf{x}) \approx -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

explaining any further assumptions that you make.

(b) A thin rod has mass m and length $2a$. It rotates non-relativistically with angular velocity ω in the xy plane. The energy-density of the rod is

$$T_{00}(t,\mathbf{x}) = \frac{m}{2a}\int_{-a}^{a} d\ell\,\delta(x-\ell\cos\omega t)\delta(y-\ell\sin\omega t)\delta(z)$$

(i) Determine $\bar{h}_{ij}$ for $r \gg a$. What is the frequency of the gravitational waves emitted by the rod? Show that on the positive z-axis, the time-dependent part of the linearized gravitational field has the same form as a gravitational plane wave propagating in the z-direction, whose amplitude is proportional to $1/z$, and whose polarization is a linear combination of $+$ and $\times$ polarizations.

[You may assume without proof $\bar{h}_{00}(t,\mathbf{x}) \approx \frac{4m}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$.*]*

(ii) Calculate the average power emitted by the rod in gravitational waves.

4

The three-dimensional *Gödel spacetime* is $(\mathbb{R}^3, g)$ where

$$g = -(dt + \sqrt{2}e^x dy)^2 + dx^2 + e^{2x} dy^2$$

where $-\infty < t, x, y < \infty$.

(a) A basis of vector fields is

$$e_0 = \frac{\partial}{\partial t} \qquad e_1 = \frac{\partial}{\partial x} \qquad e_2 = e^{-x}\frac{\partial}{\partial y} - \sqrt{2}\frac{\partial}{\partial t}$$

Show that this basis is orthonormal and that the dual basis of 1-forms is

$$e^0 = dt + \sqrt{2}e^x dy \qquad e^1 = dx \qquad e^2 = e^x dy$$

(b) The connection 1-forms are determined by $de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$. Show that

$$\omega_{01} = Ae^2 \qquad \omega_{02} = Be^1 \qquad \omega_{12} = Ce^0 + De^2$$

for certain constants A, B, C, D whose values you should determine.

(c) Calculate the curvature 2-forms using $\Theta_{\mu\nu} = d\omega_{\mu\nu} + \omega_{\mu\rho} \wedge \omega^\rho{}_\nu$. Hence calculate the Riemann tensor components using $\Theta_{\mu\nu} = \frac{1}{2}R_{\mu\nu\rho\sigma}e^\rho \wedge e^\sigma$.

(d) Calculate the Ricci tensor. Hence show that this spacetime satisfies Einstein's equation with a cosmological constant Λ where the matter is a pressureless perfect fluid: $T_{ab} = \rho u_a u_b$ with velocity $u^a = e_0^a$. You should give expressions for Λ and ρ.

(e) Write down two Killing vector fields of the above metric. Show that the map $\phi_s : (t, x, y) \mapsto (t, x + s, e^{-s}y)$ defines a 1-parameter group of isometries. Hence find another Killing vector field. Show that any point can be mapped to any other point by an isometry.

END OF PAPER