UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 9:00 am to 12:00 pm

PAPER 307

SUPERSYMMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Superspace covariant derivatives are defined by

$$\mathcal{D}_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \quad \text{and} \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

Show that

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu \quad \text{and} \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 0$$

where $\mathcal{P}_\mu$ is a differential operator that you should define.

Define a *chiral superfield*. Define an *anti-chiral superfield*.

Use the shifted coordinate

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

to determine the component expansion of a chiral superfield Φ.

Explain why one can construct a supersymmetric action by integrating a superpotential $W(\Phi)$ over only half of superspace.

A superpotential is given by

$$W_{\text{tree}} = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3$$

Explain why this superpotential is not renormalised. Why does this argument does not protect the physical mass m_{phys} and coupling λ_{phys} from renormalisation?

2 A real superfield V has component expansion

$$\begin{aligned} V(x,\theta,\bar\theta) &= C + \theta\chi + \bar\theta\bar\chi + i\theta^2 M - i\bar\theta^2 M^\dagger + \theta\sigma^\mu\bar\theta\, A_\mu \\ &\quad + \theta^2\bar\theta\left(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right) + \bar\theta^2\theta\left(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi\right) + \frac{1}{2}\theta^2\bar\theta^2\left(D - \frac{1}{2}\Box C\right) \end{aligned}$$

A chiral multiplet has the expansion,

$$\Omega = \omega + \sqrt{2}\theta\psi + \theta^2 F + i\theta\sigma^\mu\bar\theta\partial_\mu\omega - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\,\sigma^\mu\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\Box\omega$$

Explain how the shift $V \to V + i(\Omega - \Omega^\dagger)$ can be used to implement gauge transformations of A_μ and remove some of the fields in V.

A superfield is defined as

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V$$

Explain briefly how a supersymmetric action can be constructed using this superfield. (You need not compute the component form of the action.)

Supersymmetric Maxwell theory has the action

$$S = \int d^4x\ \left[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{e^2}\lambda\sigma^\mu\partial_\mu\bar\lambda\right]$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Show that the action is invariant under the supersymmetry transformations

$$\delta A_\mu = \epsilon\sigma_\mu\bar\lambda + \lambda\sigma_\mu\bar\epsilon\ , \qquad \delta\lambda = F_{\mu\nu}\ \sigma^{\mu\nu}\epsilon$$

where ϵ is a Grassmann-valued Weyl spinor and $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$.

[You may use the sigma-matrix identity $\sigma^\nu\bar\sigma^\mu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho + \eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu + i\epsilon^{\nu\mu\rho\kappa}\sigma_\kappa$.]

3 Briefly describe the implications if a symmetry has:

- a gauge anomaly,
- a chiral, or ABJ, anomaly,
- a 't Hooft anomaly.

A (non-supersymmetric) $SU(N)$ gauge theory is coupled to a single left-handed Weyl fermion λ in the anti-symmetric $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ representation and p left-handed Weyl fermions ψ^i, $i = 1, \ldots, p$, each in the anti-fundamental $\overline{\Box}$ representation.

(i) For what value of p is the quantum theory consistent?

(ii) Write down the classical, global symmetries of the theory. Show that the quantum theory has a $SU(p) \times U(1)$ global symmetry.

(iii) Compute the $SU(p)^3$, $SU(p)^2U(1)$, $U(1)^3$ and mixed $U(1)$-gravitational 't Hooft anomalies for this theory.

It is conjectured that this theory confines without spontaneously breaking any global symmetry. The massless degrees fields are thought to be a collection of gauge singlet fermions

$$\chi^{ij} = \psi^{(i}(\lambda\psi^{j)}) \quad i, j = 1, \ldots, p$$

transforming in the symmetric $\Box\!\Box$ representation of the $SU(p)$ global symmetry.

(iv) Show that the 't Hooft anomalies of the fermion χ match those of the original gauge theory.

[*Note: You may use the fact that the dimension, Dynkin index I and anomaly coefficient A of various $SU(N)$ representations are given by:*

R	$\Box$	$\Box\!\Box$	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$
$\dim(R)$	N	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$N+2$	$N-2$
$A(R)$	1	$N+4$	$N-4$

.]

4 $Sp(N_c)$ supersymmetric gauge theory, minimally coupled to $2N_f$ chiral multiplets Φ^i in the fundamental representation, where $i = 1, \ldots, 2N_f$, has the following classical symmetries:

	$Sp(N_c)$	$SU(2N_f)$	$U(1)_A$	$U(1)_{R'}$
Φ	$\square$	$\square$	1	0

(i) Show that there is a non-anomalous R-symmetry under which the gluinos have charge $+1$ and the chiral multiplets have charge $R[\Phi] = (N_f - N_c - 1)/N_f$.

(ii) For what value of N_f does the theory cease to be asymptotically free?

(iii) Use the relation between R-charge and scaling dimension to show that the meson $\Phi^i\Phi^j$ has dimension 2 at this point.

(iv) For what value of N_f does the meson $\Phi^i\Phi^j$ have the dimension of a free scalar? Hence determine the expected range of the conformal window.

It is conjectured that this theory has a dual description given by $Sp(N_f - N_c - 2)$ gauge theory coupled to $2N_f$ chiral multiplets q_i and a collection of singlet chiral multiplets M^{ij}. The chiral multiplets interact through the superpotential $W \sim q_i M^{ij} q_j$. The classical symmetries of this theory are:

	$Sp(N_f - N_c - 2)$	$SU(2N_f)$	$U(1)_A$	$U(1)_{R'}$
q	$\square$	$\overline{\square}$	1	0
M	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	-2	$+2$

(v) Determine the non-anomalous R-charge of q.

(vi) Determine the R-charge and dimension of the singlets M at the fixed point and show that they coincide with those of the meson of the original theory.

(vii) For what values of N_f is the theory no longer asymptotically free? Explain the relevance of this for the duality.

(viii) Compute the $SU(2N_f)^3$ 't Hooft anomaly in both the original theory and the dual.

[*The beta function for a supersymmetric gauge theory has one-loop coefficient*

$$b_0 = \frac{3}{2}I(\text{adj}) - \frac{1}{2}\sum_{\text{chirals}} I(R)$$

You may use the following group theoretic facts about representations of $Sp(N)$*:* $\dim(\square) = 2N$ *and* $I(\square) = 1$*;* $\dim(\text{adj}) = N(2N+1)$ *and* $I(\text{adj}) = 2(N+1)$*. You may also use the following group theoretic facts about the anti-symmetric representation of* $SU(N)$*:* $\dim(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = \frac{1}{2}N(N-1)$, $I(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = N-2$ *and* $A(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = N-4$.]

END OF PAPER