UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 9:00 am to 12:00 pm

PAPER 306

STRING THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the action

$$S[X] = -\frac{1}{4\pi\alpha'} \int \eta^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}\, d^2\sigma$$

where $\sigma^\alpha = (\tau, \sigma) \cong (\tau, \sigma + 2\pi)$ are coordinates on the worldsheet cylinder, and where the field X describes a map to $\mathbb{R}^{1,25}$ with its flat Minkowski metric.

a) Derive the classical mode expansion of $X^\mu(\tau, \sigma)$.

b) Obtain conditions these modes must satisfy if your solution is to describe a classical string.

c) Now suppose the target space is $\mathbb{R}^{1,24} \times S^1$, where the circle has circumference $2\pi R$. What periodicity condition must be imposed on the field $X^{25}(\tau, \sigma) \equiv Y(\tau, \sigma)$ that describes maps to this target space circle? Give the new form of the mode expansion for Y.

d) Show that the mass M of string states in the quantum theory is given by

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2a\right)$$

where you should explain the meaning of the integers $n, w \in \mathbb{Z}$ and $N, \tilde{N} \in \mathbb{N}_0$, and the origin of the constant a (which you need not determine).

2 A $\beta\gamma$-*system* is a pair of complex bosons with chiral action

$$S[\beta,\gamma] = \frac{1}{2\pi}\int_\Sigma \beta\bar\partial\gamma \; d^2z\,,$$

where $\bar\partial = \partial_{\bar z}$. The holomorphic stress tensor of this theory is given by

$$T(z) = -\,{:}\beta\partial\gamma{:}\,(z) + h\,{:}\partial(\beta\gamma){:}\,(z)\,,$$

where $\partial = \partial_z$ and h is a constant.

a) Derive the OPE

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w}$$

and explain why $\beta(z)\beta(w)$ and $\gamma(z)\gamma(w)$ are both non-singular.

b) Show that γ is a primary field with holomorphic conformal weight h, while β is a primary field with holomorphic conformal weight $1-h$.

c) State the form of the $T(z)T(w)$ OPE in a general 2d CFT. Calculate the holomorphic central charge associated to the stress tensor of the $\beta\gamma$ system above and show it is unchanged if we replace h by $1-h$. How would your result be different if β and γ were fermionic?

d) In a certain string theory, the (gauge-fixed) worldsheet action is

$$S = \frac{1}{2\pi\alpha'}\int_\Sigma \partial X_\mu\bar\partial X^\mu\, d^2z + \left[\frac{1}{2\pi}\int_\Sigma\left(b\bar\partial c + m\bar\partial n + \sum_{a=1}^{D/2}\chi_a\bar\partial\psi^a + \sum_{r=1}^{2}\beta_r\bar\partial\gamma^r\right)d^2z \;+\; \text{c.c.}\right].$$

Here, X^μ are the usual free scalars describing a map $X:\Sigma\to\mathbb{R}^{1,D-1}$ where D is even, and bc are the usual fermionic ghosts for worldsheet diffeomorphisms. In addition, n is a fermion with $h=0$, each ψ^a is a fermion with $h=1/2$, and each γ^r is a boson with $h=-1/2$. Using your result from part c), calculate the critical dimension of this theory.

3

a) What conditions must the operator $\mathcal{O}(X)$ obey if $c\tilde{c}\,\mathcal{O}(X)$ is to be a valid (fixed) vertex operator in the closed bosonic string?

b) *Briefly* outline why the spectrum of the closed bosonic string contains an infinite tower of particles whose mass M and spin J are related by $J = 2 + \alpha' M^2/2$. [*Recall that* $:e^{ik\cdot X}:$ *has conformal weights* $h = \tilde{h} = \alpha' k^2/4$.]

The worldsheet correlation function of three graviton vertex operators

$$U_{k,\epsilon}(z) = c\tilde{c}\ :\epsilon_{\mu\nu}\,\partial_z X^\mu\,\partial_{\bar{z}} X^\nu\,e^{ik\cdot X}:(z)$$

in the closed bosonic string at genus zero yields the three graviton tree-level amplitude

$$\mathcal{A}(k_1,k_2,k_3) = 4\pi G_{\mathrm{N}}\,(2\pi)^{26}\,\delta^{26}(k_1+k_2+k_3)\;\epsilon^{(1)}_{\mu\nu}\epsilon^{(2)}_{\kappa\lambda}\epsilon^{(3)}_{\rho\sigma}\,T^{\mu\kappa\rho}(k_i)\,T^{\nu\lambda\sigma}(k_i)$$

in a flat background space-time, where G_{N} is the Newton constant, $\epsilon^{(i)}_{\mu\nu}$ is the polarization tensor of the i^{th} graviton, and the tensor

$$T^{\mu\kappa\rho}(k_i) = k^\mu_{23}\eta^{\kappa\rho} + k^\kappa_{31}\eta^{\mu\rho} + k^\rho_{12}\eta^{\mu\kappa} + \frac{\alpha'}{8}k^\mu_{23}k^\kappa_{31}k^\rho_{12}$$

with $k^\mu_{ij} = k^\mu_i - k^\mu_j$.

c) Why is this worldsheet correlation function independent of the insertion points of the $U_{k_i,\epsilon_i}(z_i)$?

d) Why does the worldsheet correlator lead to an amplitude that is proportional to two powers of the tensor $T^{\mu\kappa\rho}$?

e) Why does $T^{\mu\kappa\rho}$ contain terms that are linear and terms that are cubic in the momenta?

f) Explain why the amplitude $\mathcal{A}(k_1,k_2,k_3)$ implies that, in bosonic string theory, gravity is *not* governed by the usual Einstein-Hilbert action

$$S[g] = \frac{1}{16\pi\,G_{\mathrm{N}}}\int R(g)\,\sqrt{-g}\,d^{26}x\,.$$

Suggest the schematic form of a target space action that is consistent with $\mathcal{A}(k_1,k_2,k_3)$. [*You do not need to worry about the detailed Lorentz index structure.*]

[*You may find it helpful to use the OPEs*

$$X^\mu(z)X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2 \qquad \textit{and} \qquad \partial_z X^\mu(z)\ :e^{ik\cdot X}:(w) \sim -\frac{i\alpha' k^\mu}{2}\,\frac{:e^{ik\cdot X}:(w)}{z-w}\,. \quad]$$

4 Consider canonical quantization of the scalar fields X^μ in the critical closed bosonic string (*i.e.* with $D = 26$ and normal ordering constant $a = 1$).

a) Explain what is meant for a state to be i) *physical*, ii) *spurious* and iii) *null*.

b) Show that a state of the form $L_{-1}|\chi\rangle$ is null if $|\chi\rangle$ obeys $L_n|\chi\rangle = 0$ for all $n \geqslant 0$.

c) Show that the state

$$\left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\phi\rangle$$

is null when $L_n|\phi\rangle = 0$ for all $n \geqslant 1$ and $L_0|\phi\rangle = -|\phi\rangle$.

d) Show that the states

$$|\psi\rangle = \left(t_{\mu\nu}\,\alpha_{-1}^\mu\alpha_{-1}^\nu + v_\mu\alpha_{-2}^\mu\right)|0,p\rangle$$

of the open string at level 2 are physical if the polarization tensors obey

$$v\cdot\alpha_0 = -\frac{1}{2}t^\mu_{\ \mu} \qquad \text{and} \qquad t_{\mu\nu}\,\alpha_0^\nu = -v_\mu\,,$$

where $\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu$ obeys $\alpha_0\cdot\alpha_0 = -2$.

e) Now consider the case where

$$v_\mu = \frac{1}{4}t^\lambda_{\ \lambda}\,\alpha_{0\mu} \qquad \text{and} \qquad t_{\mu\nu} = \frac{1}{20}(3\alpha_{0\mu}\alpha_{0\nu} + \eta_{\mu\nu})t^\lambda_{\ \lambda} + \epsilon_{\mu\nu}\,,$$

where $\epsilon_{\mu\nu}$ is a traceless symmetric tensor obeying $\epsilon_{\mu\nu}\alpha_0^\nu = 0$. Verify that this case obeys the constraints in part d). Show that

$$|\psi\rangle = \epsilon_{\mu\nu}\,\alpha_{-1}^\mu\alpha_{-1}^\nu|0,p\rangle + |n\rangle$$

where $|n\rangle$ is null. [*Hint: relate $|n\rangle$ to the state in part c).*]

[*You may use without proof the oscillator algebra*

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\eta^{\mu\nu}\,\delta_{m+n,0}\,,$$

the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\,\delta_{m+n,0}$$

and the algebra

$$[L_m, \alpha_n^\nu] = -n\,\alpha_{m+n}^\nu$$

between Virasoro generators and the oscillator modes.]

END OF PAPER