UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 1:30 pm to 4:30 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) A particle of mass m moves freely in one dimension. The classical Lagrangian is given by $L = \frac{1}{2}m\dot{x}^2$ where $x(t) \in \mathbb{R}$, and the corresponding quantum Hamiltonian is $\hat{H} = \hat{p}^2/2m$.

(i) Using operator methods, determine the probability amplitude for the particle to travel from an initial position x_i at $t = 0$ to a final position x_f at time $t = T$. We call this amplitude the *free propagator*. [Recall $\int_{-\infty}^{\infty} dy\, e^{-ay^2} = \sqrt{\pi/a}$.]

(ii) Find the trajectory $x(t)$ which minimizes the classical action, and use it to derive the free propagator (up to a position-independent, multiplicative factor).

(b) A particle of mass m moves freely along a circle of radius R. The classical Lagrangian is given by $L = \frac{1}{2}mR^2\dot{\theta}^2$, where θ the angle corresponding to the particle's position. The corresponding quantum system has energy eigenstates $|n\rangle$ with energy eigenvalues $E_n = n^2\hbar^2/2mR^2$. You can take as given that the normalized, energy eigenfunctions are $\langle\theta|n\rangle = \frac{1}{\sqrt{2\pi}}e^{in\theta}$.

(i) Using operator methods, determine the probability amplitude, or propagator, for the particle to travel from an initial angle θ_i at $t = 0$ to a final angle θ_f at time $t = T$. You may leave the answer in the form of an infinite sum over the energy eigenvalues of the system.

(ii) Find all classical paths satisfying the initial and final conditions above and evaluate the classical action of each path. Using the identity

$$\sum_{j=-\infty}^{\infty} \tilde{f}(\phi + 2\pi j) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2\pi}\right) e^{in\phi}$$

where $\tilde{f}(\kappa) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{2i\pi\kappa x}$ [equivalently, $f(x) = \int_{-\infty}^{\infty} d\kappa\, \tilde{f}(\kappa) e^{-2i\pi\kappa x}$], relate the sum over the classical paths considered above to the propagator.

(c) Without performing any explicit calculation, describe how the derivations above would need to be changed if there were a nontrivial potential, e.g. $V(x)$ in part (a) or $V(\cos\theta)$ in part (b).

2

(a) The classical action of a real scalar field $\phi(x)$ in 4 Euclidean dimensions is given by

$$S_a = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]$$

where $0 < \lambda \ll 1$. Write down the momentum-space Feynman rules for this theory, with a brief justification for each rule.

Let $V_1^{(4)}(p_1, p_2, p_3, p_4)$ be the 4-point vertex function, i.e. the sum of one-particle irreducible (1PI) diagrams with 4 external legs and 1 loop. Draw the Feynman graphs which contribute to $V_1^{(4)}(p_1, p_2, p_3, p_4)$. Using the Feynman rules, write down integral expressions for these diagrams. Introduce an explicit momentum cutoff Λ, evaluate the integral, and determine whether the integrals converge or diverge as $\Lambda \to \infty$.

Take as given that we can infer a physical value λ_{phys} from experiment. What is the renormalized coupling, λ_a in the on-shell scheme? [Hint: We expect $0 < \lambda_a \ll 1$.]

(b) Let us introduce heavy ghost fields, i.e. spinless, Grassmann-valued fields $\eta(x)$ and $\bar{\eta}(x)$ with mass $M \gg m$, by taking our action to be $S = S_a + S_b$, where

$$S_b = \int d^4x \left[c_1\partial_\mu\bar{\eta}\,\partial^\mu\eta + c_2M^2\bar{\eta}\eta + c_3\lambda\bar{\eta}\phi^2\eta\right] .$$

Write down the additional Feynman rules coming from S_b. Choose values for the numerical coefficients c_1, c_2, and c_3 so that the ghost propagator is equal to the scalar propagator (if we were to replace M by m), and so that the ghost-scalar vertex is equal to the ϕ^4 vertex.

In the theory with action S, use the Feynman rules to determine $V_1^{(4)}(p_1, p_2, p_3, p_4)$, again the one-loop vertex function with 4 external ϕ legs. What is the on-shell renormalized coupling, λ_b, in this theory? [Hint: We expect $0 < \lambda_b \ll 1$.]

(c) Assume that we are interested in calculating scattering amplitudes, such as $2 \to 2$ scattering of scalar particles, where all initial and final momenta are small compared to Λ and M. Will predictions using the theories in parts (a) and (b) differ? Why or why not?

[*Hint: You may find the following identities useful:*

$$\text{I.} \quad \int \frac{d^4k}{(2\pi)^4}\, f(k^2) = \frac{1}{16\pi^2}\int dk^2\, k^2 f(k^2)$$

$$\text{II.} \quad \int_A^B \frac{ds}{(k^2+s)^n} = -\frac{1}{n-1}\,\frac{1}{(k^2+s)^{n-1}}\bigg|_A^B \ . \ ]$$

3

In Euclidean spacetime, the QED Lagrangian is

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\not{D} + m)\psi$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$.

(a) Show that the Lagrangian is invariant under global U(1) transformations of the form $\psi(x) \mapsto e^{i\alpha}\psi(x)$, $\bar{\psi}(x) \mapsto \bar{\psi}(x)e^{-i\alpha}$, and $A_\mu(x) \mapsto A_\mu(x)$. Use this invariance to show that the current $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ is conserved.

(b) Derive the Schwinger–Dyson equation

$$\partial_\mu\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = -\left[\delta^{(4)}(x - x_1) - \delta^{(4)}(x - x_2)\right]\langle\psi(x_1)\bar{\psi}(x_2)\rangle\,.$$

(c) Use the result of (b) to derive the Ward–Takahashi identity involving the vertex function V_3^μ and the fermion propagator $G(\not{p})$:

$$iq_\mu V_3^\mu(q, p_1, p_2) = ie[G^{-1}(\not{p}_1) - G^{-1}(\not{p}_1 - \not{q})]\,.$$

(d) Explain what the Ward–Takahashi identity implies for renormalization of terms in the Lagrangian, for example in the on-shell renormalization scheme. Comment on what, if any, results could hold nonperturbatively.

4

Let $A^a_\mu(x)$ be the N^2-1 components of a gauge field $A_\mu(x)$ in $\mathrm{SU}(N)$ gauge theory. Under an infinitesimal gauge transformation parametrized by $\alpha(x) = \alpha^a(x)T^a$, $A_\mu(x) \mapsto A_\mu(x) + \delta A_\mu(x)$, where

$$\delta A_\mu(x) = \frac{1}{g}\partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)] =: \frac{1}{g} D_\mu \alpha(x)$$

Here g is the coupling, and T^a are the Hermitian generators of $\mathfrak{su}(N)$ satisfying $[T^a, T^b] = if^{abc}T^c$.

(a) How does the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ transform under this gauge transformation? Show that the Yang–Mills Lagrangian $\mathcal{L}_{\mathrm{YM}} = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}$ is gauge-invariant.

(b) Introducting anticommuting fields $c^a(x)$ and $\bar{c}^a(x)$, as well as a real field $B^a(x)$, define an operator s which acts on the fields in the following way

$$sA_\mu = D_\mu c \qquad\qquad sB = 0$$
$$sc = \frac{ig}{2}[c, c] \qquad\qquad s\bar{c} = B\,.$$

and which obeys the rule $s(XY) = (sX)Y \pm X(sY)$, with the $+$ sign if X is commuting and $-$ if X is anticommuting. Show that s is a nilpotent operator, i.e. that s^2 acting on any of the fields $\Phi \in \{A_\mu, c, \bar{c}, B\}$ yields $s^2\Phi = 0$.

(c) Consider the Lagrangian

$$\mathcal{L} = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + s\left(\bar{c}^a L[A^a] - \frac{\xi}{2}\,\bar{c}^a B^a\right)$$

where L is a linear operator, in particular $sL[A^a] = L[sA^a]$. Does $s\mathcal{L} = 0$?

By applying the s operator, put $\mathcal{L}$ in a more familiar form.

(d) Consider the two cases: (i) $L[A^a] = \partial^\mu A^a_\mu$ and (ii) $L[A^a] = n^\mu A^a_\mu$, where n^μ is a constant unit vector. In each case, explain the meaning of each term in $\mathcal{L}$ and draw the one-loop Feynman diagrams which contribute (non-vanishingly) to the gauge field propagator. [You do not need to explicitly determine any propagators or vertices.]

END OF PAPER