UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2022 9:00 am to 11:00 am

PAPER 303

STATISTICAL FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a theory which has an effective free energy density,

$$f(T,m) = \alpha_2(T)\, m^2 + \alpha_{2n}\, m^{2n}\,,$$

in the mean field approximation, where m is the magnetisation, $n > 1$ is an integer, $\alpha_{2n} > 0$ and $\alpha_2(T)$ varies from positive to negative as the temperature T is lowered, with $\alpha_2(T) \sim T - T_c$ for T close to T_c.

(i) Sketch $f(T,m)$ vs m for high and low temperatures. Determine the equilibrium magnetisation and show that there is a continuous phase transition at $T = T_c$.

(ii) Using the mean field approach, determine the critical exponents α, β, δ and γ at this phase transition.

(iii) An improved description can be obtained by allowing the magnetisation, $m(\boldsymbol{x})$, to vary with position. Write down an appropriate effective free energy functional $F[m]$.

(iv) Close to the Gaussian fixed point, for $d < 4$, fluctuations give a contribution to the heat capacity $\sim |T - T_c|^{\frac{d}{2}-2}$. Show that mean field contributions to the heat capacity dominate over these fluctuations when $d > d_c$ for some d_c that you should determine.

(b) Consider a model defined on a square lattice in d dimensions. The N lattice sites are labelled by $i \in \Lambda$ and the spin variable s_i at site i takes the value 1 or -1. The lattice has two types of sites in a checkerboard arrangement: "even" sites $i \in \Lambda_{\mathrm{e}}$ and "odd" sites $i \in \Lambda_{\mathrm{o}}$, with $\Lambda = \Lambda_{\mathrm{o}} \cup \Lambda_{\mathrm{e}}$. An even site only has odd sites as nearest neighbours and, similarly, an odd site only has even sites as nearest neighbours. You may assume that half the sites are even and half are odd. The energy is given by

$$E = J\sum_{\langle ij\rangle} s_i s_j - h\sum_{i\in\Lambda} s_i - g\sum_{i\in\Lambda_{\mathrm{e}}} s_i + g\sum_{i\in\Lambda_{\mathrm{o}}} s_i$$

where $J > 0$, g and h are constants, and $\langle ij\rangle$ means that the sum is over nearest neighbour pairs.

(i) Using the mean field approach, derive an expression for the effective free energy per site, $f(\beta, m_{\mathrm{e}}, m_{\mathrm{o}})$, where $m_{\mathrm{e}} = \frac{2}{N}\sum_{i\in\Lambda_{\mathrm{e}}} s_i$, $m_{\mathrm{o}} = \frac{2}{N}\sum_{i\in\Lambda_{\mathrm{o}}} s_i$ and $\beta = \frac{1}{T}$.

(ii) Hence show that equilibrium values of m_{e} and m_{o} satisfy,

$$m_{\mathrm{e}} = \tanh\left[\beta A(m_{\mathrm{o}})\right]\,, \qquad m_{\mathrm{o}} = \tanh\left[\beta D(m_{\mathrm{e}})\right]\,,$$

where you should specify the functions $A(m_{\mathrm{o}})$ and $D(m_{\mathrm{e}})$.

(iii) Consider $h = g = 0$, expand f to second order in m_{e}, m_{o}, and then write f in terms of $m_{\pm} = \frac{1}{2}(m_{\mathrm{e}} \pm m_{\mathrm{o}})$. Show that there is a phase transition for m_- and determine the critical temperature. Is there a phase transition for m_+? [You may assume that the coefficients of $\mathcal{O}(m^4)$ terms are positive.]

2

Consider a theory involving a real scalar field ϕ in d dimensions with a free energy of the form

$$F[\phi] = \int d^d x \Big[\tfrac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + \tfrac{1}{2}\mu_0^2 \, \phi^2 + \ldots \Big] \, . \qquad (*)$$

(a) Describe the three steps of the renormalisation group procedure (in momentum space) for such a free energy and explain how they result in a flow of the parameters in the free energy. You should denote the original cutoff Λ and the new cutoff Λ/ζ.

(b) Now suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with

$$\sim \int d^d x \Big[\lambda_0 \, \phi^3 + g_0 \, \phi^4 + \gamma_0 \, \phi^6 \Big] \, .$$

Calculate the naive (engineering) dimensions of ϕ, μ_0, λ_0, g_0 and γ_0. State whether each of the couplings is relevant, marginal or irrelevant.

(c) Draw Feynman diagrams representing the corrections to the coupling γ_0 from the $g_0 \, \phi^4$ term up to and including order g_0^3. Calculate these corrections and give an expression for the flow of the coupling, $\gamma(\zeta)$. [You may assume that $\langle \phi^+_{\boldsymbol{k}} \, \phi^+_{\boldsymbol{k}'} \rangle_+ = (2\pi)^d \, \delta^{(d)}(\boldsymbol{k} + \boldsymbol{k}') G_0(k)$, where $G_0(k) = 1/(k^2 + \mu_0^2)$, for appropriately defined ϕ^+ and $\langle \ldots \rangle_+$, you may use Wick's theorem without proof, and you may leave your final answer in integral form. You may ignore corrections from $\lambda_0 \, \phi^3$ and $\gamma_0 \, \phi^6$ terms, and the rescaling of the field.]

3

Consider an $O(N)$ model involving an N-component real field $\boldsymbol{\phi}(\boldsymbol{x})$ in d dimensions with free energy,

$$F_1[\boldsymbol{\phi}] = \int d^d x \left[\tfrac{1}{2}(\partial_i \phi_a)(\partial_i \phi_a) + \tfrac{1}{2}\mu_0^2\, \boldsymbol{\phi}\cdot\boldsymbol{\phi} + g_0\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 \right],$$

where repeated indices are summed over, $i = 1, 2, \ldots, d$ and $a = 1, 2, \ldots, N$.

(a) Draw the Feynman diagrams representing the leading order corrections to (i) the coupling μ_0^2 from the g_0 term and (ii) the coupling g_0 from the g_0 term.

(b) Given that in the Ising model,

$$\begin{aligned} \mu^2(\zeta) &= \zeta^2 \left(\mu_0^2 + 12\, g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \right), \\ g(\zeta) &= \zeta^{4-d} \left(g_0 - 36\, g_0^2 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2} \right), \end{aligned}$$

derive analogous results for the $O(N)$ model. [You may leave your answers in integral form.]

(c) Derive the β functions, $\frac{d\mu^2}{ds}$ and $\frac{dg}{ds}$, where $s = \ln\zeta$. [You may use Ω_{d-1} to denote the area of the unit sphere S^{d-1}.]

(d) Consider $d = 4 - \epsilon$ and define $\tilde{g} = \Lambda^{-\epsilon} g$. Working to leading order in ϵ, find the fixed points and calculate the critical exponent ν at each fixed point.

(e) Now suppose that

$$F[\boldsymbol{\phi}] = F_1[\boldsymbol{\phi}] + \int d^d x\, \lambda_0 \sum_{a=1}^{N} (\phi_a)^4 .$$

Draw Feynman diagrams for the corrections to the couplings (i) g_0 and (ii) λ_0 proportional to λ_0^2 and $\lambda_0 g_0$. Indicate which of these diagrams (if any) are proportional to N.

END OF PAPER