UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, you may quote results from lectures that you use for the calculation without proof, provided you clearly state them.

Take a real simple Lie algebra $L(G)$ with exactly two simple roots

$$\boldsymbol{\alpha}_1 = (1,\ 0), \qquad \boldsymbol{\alpha}_2 = (-1,\ 1)\,.$$

Assume that $E_{i\pm}, H_i$ are the $SU(2)$ generators associated with root $\boldsymbol{\alpha}_i$ $(i = 1, 2)$.

(i) Calculate the Cartan matrix. Then calculate the weight space of the adjoint representation, writing each weight in terms of the $\boldsymbol{\alpha}_i$. Draw its weight diagram.

(ii) Determine the fundamental weights $\boldsymbol{\mu}_i$ explicitly, in terms of the $\boldsymbol{\alpha}_i$.

(iii) By using $E_{i\pm}$, find weights for the irreducible representation d with highest weight $\boldsymbol{\mu}_2$, writing each weight in terms of $\boldsymbol{\mu}_2$ and the $\boldsymbol{\alpha}_i$. Draw the corresponding weight diagram. Is the representation real or complex? You may assume that the weights all have multiplicity 1.

(iv) Which complex Lie algebra in Cartan's classification corresponds to the complexification of $L(G)$ (note any isomorphisms)? Given that the representation identified in (iii) is in the fundamental representation of the group, identify G. Putting an appropriate scalar field ϕ in the fundamental representation space of G, write its transformation under a gauge transformation $g(x) \in G$. Write the gauge transformation of the associated Yang-Mills gauge field A_μ. How many real degrees of freedom do you expect in the second component, A_1?

(v) In terms of the gauge coupling λ, define a covariant derivative $D_\mu \phi$ and compute its gauge transformation.

(vi) Defining the non-abelian field strength tensor $F_{\mu\nu}$ by $F_{\mu\nu}\phi = \frac{1}{\lambda}[D_\mu, D_\nu]\phi$, compute the field strength solely in terms of A_μ and partial derivatives.

(vii) By calculating the transformation of $F_{\mu\nu}\phi$, derive the transformation of $F_{\mu\nu}$ in terms of $g(x)$.

(viii) Write a general Lorentz invariant, renormalisable, *locally* $G-$invariant Lagrangian density $\mathcal{L}$ for ϕ. Give the mass dimensions of any constants that you introduce.

(ix) Verify the local $G-$invariance of $\mathcal{L}$.

2 A Poincaré transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ is written here as (Λ, a), where $\Lambda^\mu{}_\nu$ is a Lorentz transformation.

(i) Compute $(L, b)(\Lambda, a)$.

(ii) Determine $(\Lambda, a)^{-1}$ in terms of Λ and a.

(iii) Defining $(\Lambda_c, a_c) := [(L, b),\ (\Lambda, a)]$, where $[,]$ denotes the commutator, find Λ_c and a_c in terms of L, Λ, b and a.

(iv) Defining the infinitesimal Lorentz transformations $\Lambda := I + \omega_\Lambda + \mathcal{O}(\omega_\Lambda^2)$ and $L := I + \omega_L + \mathcal{O}(\omega_L^2)$, find a_c to first order in ω_L and first order in ω_Λ.

(v) Putting $\Lambda_c := I + \omega_c + \mathcal{O}(\omega_c^2)$ and neglecting terms of $\mathcal{O}(\omega_L^2)$ and those of $\mathcal{O}(\omega_\Lambda^2)$, find Λ_c in terms of ω_L and ω_Λ.

(vi) Write the defining relation for $\Lambda^\mu{}_\nu$ in terms of the Minkowski metric $\eta_{\rho\sigma}$ and thus derive a symmetry relation for ω_Λ under interchange of its indices.

(vii) We now introduce a unitary operator corresponding to a Poincaré transformation $U[(\Lambda, a)]$, imposing that

$$U[(\Lambda_2,\ a_2)]U[(\Lambda_1,\ a_1)] = U[(\Lambda_2,\ a_2)(\Lambda_1,\ a_1)].$$

Expanding in infinitesimal ω and a parameters $U[(\Lambda(\omega), a)] := I + \frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + a^\mu P_\mu + \ldots$, where $M_{\mu\nu} = -M_{\nu\mu}$ and P_μ are operators. Retaining constant, linear and bilinear terms in ω_Λ, ω_L, a, b but neglecting those of quadratic order, show that

$$U[(\Lambda_c(\omega_c), a_c)] = I + \left[\frac{1}{2}\omega_L{}^{\mu\nu}M_{\mu\nu} + b^\mu P_\mu,\ \frac{1}{2}\omega_\Lambda{}^{\rho\sigma}M_{\rho\sigma} + a^\rho P_\rho\right] + \ldots$$

(viii) Thus derive the Poincaré algrebra, i.e. $[P_\mu,\ P_\nu]$, $[M_{\mu\sigma},\ P_\rho]$ and $[M_{\mu\nu},\ M_{\rho\sigma}]$.

3

(a) The Pauli matrices σ_i are defined to be

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we define I to be the 2 by 2 identity matrix. By considering that $\{I, \sigma_i\}$ where $i \in \{1,2,3\}$ form a basis of complex matrices and by using the simultaneous transformations $\sigma_i \to R_{ji}\sigma_j$, $I \to I$, where $R \in SO(3)$, show that

$$\sigma_j \sigma_k = I\delta_{jk} + i\epsilon_{jkl}\sigma_l.$$

(b) Take a theory invariant under the following simultaneous global symmetry transformations of complex scalar fields ϕ_1 and ϕ_2:

$$\begin{aligned} \phi_1 &\to \left(\cos z + i\sqrt{1-x^2-y^2}\sin z\right)\phi_1 - (y\sin z + ix\sin z)\,\phi_2, \\ \phi_2 &\to (y\sin z - ix\sin z)\,\phi_1 + \left(\cos z - i\sqrt{1-x^2-y^2}\sin z\right)\phi_2, \end{aligned}$$

where x, y, z are arbitrary real parameters satisfying $x^2 + y^2 \leqslant 1$:

(i) Identify the corresponding matrix group and the representation space of complex scalar fields that it acts upon. Give full justification.

(ii) Write down the most general renormalisable Lorentz invariant Lagrangian density that involves only ϕ_1, ϕ_2 but respects the symmetries given.

(iii) Draw Feynman diagrams of any interactions involving ϕ_1 and/or ϕ_2 and write their respective Feynman rules.

4

(i) Define a representation D of a group G over a field F.

(ii) Define a representation d of Lie algebra $L(G)$.

(iii) For $t \in \mathbb{R}$ and square matrices A and B, a truncated Baker-Campbell-Hausdorff (BCH) formula states that

$$\exp(tA)\ \exp(tB) = \exp(\alpha(A,B) + t\beta(A,B) + t^2\gamma(A,B) + \mathcal{O}(t^3)). \quad (*)$$

By expanding in t, find the functions α, β and γ.

(iv) Show that $D(g = \exp X) = \exp(d(X))$ is a representation of the image of the exponential of $L(G)$. Explicitly state how each line in your proof is obtained. You may use properties of the higher order terms in the BCH formula that you know without proof, provided that you state them clearly at any points they are used.

(v) Let A be an $n \times n$ matrix over the complex numbers and $U = \exp A$ be a non-singular matrix. Show that $A^\dagger = -A \Rightarrow U \in U(n)$.

(vi) Identify the set $L_{\mathbb{R}}(U(n))$ and thus calculate the real dimension of $L_{\mathbb{R}}(U(n))$.

END OF PAPER