UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A quantum system has a Hamiltonian of the form,

$$\hat{H} \;=\; \hat{H}_0 + \lambda \hat{H}_{int}$$

where the $\hat{H}_0$ is the Hamiltonian of a free system whose eigenstates are known and λ is a coupling constant. Define the *interaction picture* and show that the time dependence of states in this picture is determined by the differential equation,

$$i\frac{d}{dt}|\psi(t)\rangle_I \;=\; \hat{H}_I(t)\,|\psi(t)\rangle_I \tag{1}$$

where you should determine the time-dependent operator $\hat{H}_I(t)$.

Write down a formal solution of (1) checking that the differential equation is obeyed up to quadratic order in an expansion in λ. A transition amplitude $\mathcal{T} :=_S \langle f|\exp(-i\hat{H}T)|i\rangle_S$ is defined between Schödinger picture initial and final states,

$$\begin{aligned} |i\rangle_S \;&:=\; |\psi_i(-T/2)\rangle_S \\ |f\rangle_S \;&:=\; |\psi_f(+T/2)\rangle_S \end{aligned}$$

Using your solution to (1), derive a formula for $\mathcal{T}$ as a matrix element ${}_I\langle f|\hat{U}|i\rangle_I$ of an operator $\hat{U}$ (which you should determine) between the corresponding interaction picture states.

Consider a field theory of two scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ with masses m_1 and m_2 (with $m_1 > 3m_2$). The two fields are coupled through an interaction term,

$$\Delta\mathcal{L} \;=\; -\lambda\varphi_1(x)\varphi_2^3(x)$$

in their Lagrangian density. Write down a formula giving the amplitude $\mathcal{A}$ for a particle of field φ_1 with momentum $\mathbf{p}_1$ to decay into three particles of field φ_2 with momenta $\mathbf{p}'_1$, $\mathbf{p}'_2$ and $\mathbf{p}'_3$. Evaluate $\mathcal{A}$ at leading order in perturbation theory in the coupling λ.

2 *[In this question you may use without proof any of the properties of the Dirac γ-matrices demonstrated in the lectures.]*

State and prove Noether's theorem for a theory of a single Dirac spinor field $\psi(x)$. [You may assume that the Lagrangian density $\mathcal{L}(x)$ at the spacetime point x is a function of $\psi(x)$, $\bar{\psi}(x)$ and their spacetime derivatives $\partial_\mu\psi(x)$ and $\partial_\mu\bar{\psi}(x)$]

Write down the Dirac action, S_{free}, for a free massless fermion.

i) Show that the massless action S_{free} is invariant under the transformation,

$$\psi(x) \quad \to \quad \lambda^{-\Delta}\psi(\lambda^{-1}x)$$

for real $\lambda \neq 0$ and for a specific value of the constant Δ which you should determine. Find the corresponding Noether current.

ii) Define the matrix $\gamma^5 := -i\gamma^0\gamma^1\gamma^2\gamma^3$ and show that,

$$\gamma^\mu \exp\left(i\alpha\gamma^5\right) \quad = \quad \exp\left(-i\alpha\gamma^5\right)\gamma^\mu$$

for any constant α. Hence show that the massless action S_{free} is invariant under the transformation,

$$\psi(x) \quad \to \quad \exp(i\alpha\gamma^5)\psi(x)$$

for real $\alpha \in [0, 2\pi]$. Find the corresponding Noether current.

[For both the symmetry transformations of $\psi(x)$ discussed above, you should give the corresponding transformation of $\bar{\psi}(x)$ explicitly.]

3 Two Dirac spinor fields $\psi(x)$ and $\chi(x)$ with masses m_ψ and m_χ interact with a real scalar field $\varphi(x)$ of mass m_φ through an interaction term in the Lagrangian of the form,

$$\Delta\mathcal{L} \;=\; -g\,\varphi(x)\left(\bar{\psi}(x)\psi(x) + \bar{\chi}(x)\chi(x)\right)$$

where g is a coupling constant. Write down momentum space Feynman rules for the scattering amplitudes of this theory, giving factors associated with internal and external lines in diagrams as well as vertices.

Using the Feynman rules, determine the leading order contribution in perturbation theory to the scattering amplitude $\mathcal{A}$ for the process,

$$\psi(p,r) \,+\, \bar{\psi}(q,s) \;\;\rightarrow\;\; \chi\left(p',r'\right) \,+\, \bar{\chi}\left(q',s'\right).$$

Here $\psi(p,r)$ denotes a particle of the field ψ with on-shell four-momentum p, and spin index $r = 1,2$, and $\bar{\psi}(...)$ similarly denotes an anti-particle.

Setting $\mathcal{A} = (2\pi)^4\delta^{(4)}(p+q-p'-q')\mathcal{M}$, the reduced amplitude $\mathcal{M} = \mathcal{M}^{r,s;r',s'}(p,q;p'q')$ contributes to quantities which can be measured in a scattering experiment via its spin-averaged square,

$$X \;:=\; \frac{1}{4}\sum_{r,s,r',s'=1,2}\left|\mathcal{M}^{r,s;r',s'}(p,q;p'q')\right|^2.$$

Evaluate X at leading order in g, giving the answer explicitly as a function of the four-momenta and masses of the particles.

[*You may use without proof the following identities for the standard basis of positive and negative frequency solutions, denoted $u^r(p)$ and $v^r(p)$ respectively, of the Dirac equation of a fermion of mass m and 4-momentum p:*

$$\sum_{r=1,2} u^r_\alpha(p)\bar{u}^r_\beta(p) \;=\; \left(\gamma^\mu p_\mu + m\mathbb{I}_4\right)_{\alpha\beta}$$

$$\sum_{r=1,2} v^r_\alpha(p)\bar{v}^r_\beta(p) \;=\; \left(\gamma^\mu p_\mu - m\mathbb{I}_4\right)_{\alpha\beta}$$

where $\mathbb{I}_4$ denotes the 4×4 unit matrix. You may also use without proof any properties of the Dirac γ-matrices you may require as long as they are clearly stated.]

4 The gauge-fixed Lagrangian for an electromagnetic field with four-vector potential $A_\mu(x)$ is,

$$\mathcal{L} \quad = \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \; - \; \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2$$

where ξ is the gauge fixing parameter. Choosing *Feynman gauge* with $\xi = 1$, find the corresponding Euler-Lagrange equation for $A_\mu(x)$. Find the the conjugate momenta $\pi^\mu(x)$ for each component of $A_\mu(x)$.

In canonical quantization, the Schrödinger picture field operator $\hat{A}_\mu(\mathbf{x})$ and conjugate momentum $\hat{\pi}_\mu(\mathbf{x}) = \eta_{\mu\nu}\hat{\pi}^\nu(\mathbf{x})$ have mode expansions,

$$\begin{aligned}
\hat{A}_\mu\left(\mathbf{x}\right) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\mathbf{p}|}} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda\left(p\right) \left[\hat{a}_{\mathbf{p}}^\lambda \exp\left(i\mathbf{p}\cdot\mathbf{x}\right) + \hat{a}_{\mathbf{p}}^{\lambda\,\dagger} \exp\left(-i\mathbf{p}\cdot\mathbf{x}\right)\right] \\
\hat{\pi}_\mu\left(\mathbf{x}\right) &= \int \frac{d^3p}{(2\pi)^3}\, i\sqrt{\frac{|\mathbf{p}|}{2}} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda\left(p\right) \left[\hat{a}_{\mathbf{p}}^\lambda \exp\left(i\mathbf{p}\cdot\mathbf{x}\right) - \hat{a}_{\mathbf{p}}^{\lambda\,\dagger} \exp\left(-i\mathbf{p}\cdot\mathbf{x}\right)\right]
\end{aligned}$$

where $\varepsilon_\mu^\lambda(p)$ are a suitable basis of orthonormal polarisation four-vectors and $\hat{a}_{\mathbf{p}}^\lambda$, $\hat{a}_{\mathbf{p}}^{\lambda\,\dagger}$ are oscillator variables with commutation relations,

$$\left[\hat{a}_{\mathbf{p}}^\lambda, \hat{a}_{\mathbf{q}}^{\rho\,\dagger}\right] \quad = \quad -\eta^{\lambda\rho}\left(2\pi\right)^3 \delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right),$$

Starting from this mode expansion, determine the commutator $[\hat{A}_\mu(\mathbf{x}), \hat{\pi}_\nu(\mathbf{y})]$.

In this gauge the Hamiltonian operator can be written as,

$$\hat{H} \quad = \quad -\frac{1}{2}\int d^3x \left[\hat{\pi}_\mu(\mathbf{x})\hat{\pi}^\mu(\mathbf{x}) + \sum_{i=1}^{3} \nabla_i \hat{A}_\mu(\mathbf{x}) \nabla_i \hat{A}^\mu(\mathbf{x})\right]$$

where $\nabla_i = \partial/\partial x_i$.

Find the normal ordered Hamiltonian $:\hat{H}:$ in terms of the oscillator variables and discuss the resulting spectrum of particle states. Briefly describe the role of gauge invariance in restricting to physical states.

END OF PAPER