UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 1:30 pm to 3:30 pm

PAPER 215

MIXING TIMES OF MARKOV CHAINS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For a Markov chain X starting at x under $\mathbb{P}_x$, give the definition of a strong stationary time T. Define also the separation distance $s(t)$ at time t. Show that

$$s(t) \leqslant \sup_x \mathbb{P}_x(T > t).$$

For each $k \in \mathbb{N}_0$, denote by $\mathbb{Z}_{2^k} (= \{0, \ldots, 2^k - 1\})$ the cycle of size 2^k, and by $(X_j^{(k)})_{j \in \mathbb{N}_0}$ the lazy simple random walk on $\mathbb{Z}_{2^k}$ started from 0. For each $j \in \mathbb{N}_0$, let $T_j^{(k)}$ be the first time $X^{(k)}$ has jumped $2j$ times (we say that a lazy walk jumps if it does not stay in the same vertex).

(b) For each $k \in \mathbb{N}_0$, show that $\big(X_{T_j^{(k)}}^{(k)}/2\big)_{j \in \mathbb{N}_0}$ has the same law as $(X_j^{(k-1)})_{j \in \mathbb{N}_0}$.

(c) Assume that τ_{k-1} is a strong stationary time for $(X_{T_j^{(k)}}^{(k)}/2)_{j \in \mathbb{N}_0}$. Show that $\tau_k := T_{\tau_{k-1}}^{(k)} + 1$ is a strong stationary time for $(X_j^{(k)})_{j \in \mathbb{N}_0}$.
You may use without a proof that $T_{\tau_{k-1}}^{(k)}$ is a stopping time independent of $X_{T_{\tau_{k-1}}^{(k)}}^{(k)}$.

(d) For each $k \in \mathbb{N}_0$, construct a strong stationary time τ_k for $(X_j^{(k)})_{j \in \mathbb{N}_0}$ with $\mathbb{E}_0[\tau_k] = (4^k - 1)/3$.
You may use without a proof that $(T_j^{(k)})_{j \in \mathbb{N}_0}$ is independent of τ_{k-1} in (c).

2 Let $G = (V, E)$ be a connected graph with n vertices and maximal degree Δ. Denote by $d(t)$ the distance to stationarity at time t associated to the Glauber dynamics for the Ising model on G with inverse temperature β, that is associated with the following probability measure on $\{-1,1\}^V$

$$\pi_\beta(\sigma) = \frac{1}{Z_\beta} \exp\left(\beta \sum_{\{v,w\}\in E} \sigma_v \sigma_w\right) \text{ for all } \sigma \in \{-1,1\}^V,$$

where Z_β is a normalizing constant chosen so that π_β is a probability measure.

(a) Give the definition of Glauber dynamic for the Ising model on G. Let $\sigma \in \{-1,1\}^V$, $v \in V$ and $x \in \{-1,1\}$, and define $\sigma' \in \{-1,1\}^V$ by $\sigma'(w) = \sigma(w)$ if $w \neq v$ and $\sigma'(v) = x$. Verify that the associated transition probability from σ to σ' is given by

$$\frac{1 + \tanh(\beta x S_v(\sigma))}{2n},$$

where $S_v(\sigma) = \sum_{w:\{w,v\}\in E} \sigma(w)$.

(b) Give the definition of the distance to stationarity $d(t)$ and state without a proof the path metric method, that is the method which uses the path metric to bound $d(t)$.

(c) Show that if $\Delta \tanh(\beta) < 1$, then for all $t \in \mathbb{N}$

$$d(t) \leqslant n\left(1 - \frac{1 - \Delta\tanh(\beta)}{n}\right)^t.$$

You may use without a proof that

$$\sup_{x\in\mathbb{R}} \tanh(\beta(x+2)) - \tanh(\beta x) = 2\tanh(\beta).$$

3

(a) Give the definition of the spectral gap associated to a reversible Markov chain.

(b) State without a proof the comparison via canonical paths technique, that is the method to compare the spectral gaps associated to two different reversible Markov chains defined on the same state space.

Fix some $d \in \mathbb{N}$ (on which all constants in this exercise may depend), denote for each $n \in \mathbb{N}$ by $\mathbb{Z}_n^d (= \{0, \ldots, n-1\}^d)$ the d-dimensional torus with size length n. Moreover, for each $n \in \mathbb{N}$ write $2\mathbb{Z}_{2n}^d = \{2x : x \in \mathbb{Z}_{2n}^d\}$ and let $(e_1, \ldots, e_d)$ be the standard basis of $\mathbb{Z}_{2n}^d$. For all connected sets $A \subset \mathbb{Z}_{2n}^d$, we denote by $\gamma^{(2n)}(A)$ the spectral gap associated to the lazy simple random walk on the graph A, with edges between x and y in A if and only if $\{x, y\}$ is an edge of $\mathbb{Z}_{2n}^d$. Finally let

$$S_n = \big\{2x, 2x + e_i, 2x - e_i, x \in \mathbb{Z}_{2n}^d, i \in \{1, \ldots, d\}\big\},$$

In other words, S_n corresponds to $2\mathbb{Z}_{2n}^d$, plus all the neighbors of $2\mathbb{Z}_{2n}^d$.

(c) Show that there exists a constant $c' > 0$, such that for all $n \in \mathbb{N}$ and connected sets $A \subset S_n$ with $2\mathbb{Z}_{2n}^d \subset A$ we have $\gamma^{(2n)}(A) \leqslant c'/n^2$.

In (c), you can use the following two facts without a proof.

- There exists constants $\tilde{c}, \tilde{c}' > 0$ such that for all $n \in \mathbb{N}$, the spectral gap $\gamma^{(n)}$ associated to the lazy simple random walk on $\mathbb{Z}_n^d$ verifies $\tilde{c}/n^2 \leqslant \gamma^{(n)} \leqslant \tilde{c}'/n^2$.
- Let G be a connected and finite graph, X be the lazy simple random walk on G with invariant distribution π and spectral gap γ, $B \subset V(G)$ be a subset of its vertex set, and $\tau_B^+ = \inf\{k \geqslant 1 : X_k \in B\}$ be the first return time in B. The chain induced by X on B is the Markov chain on B with transition matrix $P_B(x, y) = \mathbb{P}_x(X_{\tau_B^+} = y)$ for all $x, y \in B$. Then, $(\pi(x)/\pi(B))_{x \in B}$ is the invariant distribution of the chain induced by X on B, and if γ_B denotes the spectral gap associated to the chain induced by X on B we have $\gamma_B \geqslant \gamma$.

4 Let G be a connected graph with vertex set V, π^G be the invariant distribution associated to the lazy simple random walk on G, and Φ_*^G be the associated bottleneck ratio. We denote by d_G the diameter of G, that is the maximal graph distance between two vertices of G. We also define $B(x,t) = \{y \in V : d(x,y) \leqslant t\}$ the ball centered at x with radius t for the graph distance d on G.

(a) Give the definition of the bottleneck ratio Φ_*^G. If γ^G denotes the spectral gap associated to the lazy simple random walk on G, show that $\gamma^G \leqslant 2\Phi_*^G$.
You may use without a proof the variational characterization of the spectral gap via the Dirichlet form.

(b) Show that for all $x \in V$ and $t, s \in \mathbb{N}_0$ such that $\pi^G(B(x,t+s)) \leqslant 1/2$ we have

$$\pi^G\big(B(x,t+s)\big) \geqslant \pi^G\big(B(x,s)\big)\big(1+\Phi_*^G\big)^t.$$

(c) Show that

$$\Phi_*^G \leqslant \big(2\pi_*^G\big)^{-\frac{4}{d_G-2}} - 1, \tag{1}$$

where

$$\pi_*^G = \min_{x \in V} \pi^G\big(B(x, \lfloor (d_G-1)/4 \rfloor)\big).$$

Deduce a lower bound on the relaxation time t_{rel}^G, associated to the lazy simple random walk on G, in terms of the diameter d_G and π_*^G.

END OF PAPER