UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 9:00 am to 12:00 pm

PAPER 211

ADVANCED FINANCIAL MODELS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a discrete-time market with n assets. At time t, asset i pays a dividend δ_t^i and its price immediately after the dividend payment is P_t^i. Assume the dividend and price processes are adapted to a filtration. Adopt the notation

$$X_t^{x,H} = \begin{cases} x & \text{if } t = 0 \\ H_t \cdot (\delta_t + P_t) & \text{if } t \geqslant 1 \end{cases}$$

and

$$C_t^{x,H} = X_t^{x,H} - H_{t+1} \cdot P_t$$

for $x \in \mathbb{R}$ and n-dimensional previsible process H.

(a) Briefly explain the economic interpretation of the processes $X^{x,H}$ and $C^{x,H}$. In terms of this notation, what is a *numéraire strategy*?

(b) Let x and H be such that $X_t^{x,H} > C_t^{x,H} \geqslant 0$ for all $t \geqslant 0$. Show that there exists a numéraire strategy.

Now suppose that there exists a numéraire strategy η and let $N_t = X_t^{\nu,\eta}$ for $t \geqslant 0$, where $\nu = \eta_1 \cdot P_0$.

(c) Given x and H, find another strategy K such that

$$C_t^{x,K} = 0$$
$$X_t^{x,K} = X_t^{x,H} + N_t \sum_{s=0}^{t-1} \frac{C_s^{x,H}}{N_s}$$

for all $t \geqslant 0$.

(d) Let

$$\tilde{P}_t = P_t + N_t \sum_{s=1}^{t} \frac{\delta_s}{N_s}$$

for all $t \geqslant 0$. Consider a new market with n new assets, where new asset i pays no dividend and has time-t price $\tilde{P}_t^i$. Adopt the notation

$$\tilde{X}_t^{x,H} = \begin{cases} x & \text{if } t = 0 \\ H_t \cdot \tilde{P}_t & \text{if } t \geqslant 1 \end{cases}$$

and

$$\tilde{C}_t^{x,H} = \tilde{X}_t^{x,H} - H_{t+1} \cdot \tilde{P}_t$$

Given x and H, find a strategy K such that

$$C_t^{x,K} = \tilde{C}_t^{x,H}$$
$$X_t^{x,K} = \tilde{X}_t^{x,H}$$

for all $t \geqslant 0$.

2

Consider a discrete-time market of N zero-coupon bonds of maturities $1, 2, \ldots, N$, each with risk-free unit payout. Let P_t^T be the time-t price of the bond with maturity T. Assume that there is no arbitrage in this market.

(a) Explain why there exists a positive adapted process $Y = (Y_t)_{t \geqslant 0}$ such that

$$P_t^T = \frac{\mathbb{E}(Y_T | \mathcal{F}_t)}{Y_t}.$$

You may use any result from the course without proof, as long as it is carefully stated.

Let

$$r_t = \frac{1}{P_{t-1}^t} - 1.$$

(b) Suppose that $r_t \geqslant 0$ for all $t \geqslant 1$. Prove that the process Y from (a) is a supermartingale.

(c) Fix $T \leqslant N$ and consider a European contingent claim with time T payout $\xi_T = r_T - f$, where f is a constant. Find a strategy of trading zero-coupon bonds that replicates this claim. Show that if

$$f = \frac{P_0^{T-1}}{P_0^T} - 1$$

then the initial no-arbitrage price of the claim is $\xi_0 = 0$.

(d) An interest rate swap is a contract that makes T payments of the amount $r_t - s$ at each time $t = 1, \ldots, T$, where s is a constant. If the initial no-arbitrage price of this claim is zero, find s in terms of the initial bond prices $P_0^1, \ldots, P_0^T$.

3

Consider a one-period market of n assets with the time-t price of asset i denoted P_t^i. No asset pays a dividend.

(a) What does it mean to say a contingent claim can be *replicated*? What does it mean to say that the market is *complete*?

Now suppose that the market is complete.

(b) Show that the sample space can be partitioned into at most n events of positive probability.

(c) Suppose that $P_0 = \mathbb{E}(XP_1) = \mathbb{E}(YP_1)$ for random variables X and Y. Show that $X = Y$ almost surely.

(d) Let Q be the $n \times n$ matrix $Q = \mathbb{E}(P_1 P_1^\top)$, where the notation $A^\top$ denotes the transpose of the matrix A. Suppose that Q is positive definite, and let $Z = P_1^\top Q^{-1} P_0$. Assuming the market has no arbitrage, show that $Z > 0$ almost surely. [You may use a fundamental theorem of asset pricing if carefully stated.]

4 Given a n-dimensional adapted process M and n-dimensional previsible process H, let

$$X_t = \sum_{s=1}^{t} H_s \cdot (M_s - M_{s-1}).$$

(a) Show that if M is a martingale, then X is a local martingale. [You may use without proof the fact that the martingale transform of a *bounded* previsible process with respect to a martingale is a martingale.]

For the rest of the problem, the assumption that M is a martingale is now dropped. Let

$$T = \inf\{t \geqslant 1 : \mathbb{P}(X_t \geqslant 0) = 1 \text{ and } \mathbb{P}(X_t > 0) > 0\}$$

and suppose $T < \infty$. Let

$$\theta = \frac{H_T}{\|H_T\|} \mathbf{1}_{\{H_T \neq 0, X_{T-1} \leqslant 0\}}$$

and $\xi = \theta \cdot (M_T - M_{T-1})$.

(b) Show that $\xi \geqslant 0$ almost surely.

(c) Show that $\mathbb{P}(\xi > 0) = \mathbb{P}(X_T > 0, X_{T-1} = 0) + \mathbb{P}(X_{T-1} < 0)$.

(d) Consider three cases:

(i) Case $\mathbb{P}(X_{T-1} = 0) = 1$.

(ii) Case $\mathbb{P}(X_{T-1} < 0) > 0$.

(iii) Case $\mathbb{P}(X_{T-1} = 0) < 1$ and $\mathbb{P}(X_{T-1} < 0) = 0$.

Show that in cases (i) and (ii) that $\mathbb{P}(\xi > 0) > 0$. Show that case (iii) is impossible.

(e) Show that M cannot be a martingale.

5

Consider a continuous-time market with n-dimensional Itô process $P = (P_t)_{t \geqslant 0}$ modelling the prices of n assets. No asset pays a dividend.

(a) What does it mean to say a n-dimensional previsible process H is a *self-financing strategy*? What does it mean to say a self-financing strategy H is *admissible*? What does it mean to say that a real-valued Itô process Y is a *local martingale deflator*?

(b) Let H be an admissible self-financing strategy and Y a local martingale deflator. Let $X_t = H_t \cdot P_t$ for all $t \geqslant 0$. Show that the process XY is a supermartingale.

(c) Fix an initial wealth $X_0 > 0$ and time horizon $T > 0$, and consider the problem of maximising the expected utility $\mathbb{E}[U(X_T)]$ over admissible self-financing strategies H where $X_t = H_t \cdot P_t$ for all $0 \leqslant t \leqslant T$, and where $U : \mathbb{R}_+ \to \mathbb{R}$ is increasing, concave and differentiable.

Let $\hat{U}(y) = \sup_{x \geqslant 0}\{U(x) - xy\}$. Show that the inequality

$$\mathbb{E}[U(X_T)] \leqslant \mathbb{E}[\hat{U}(Y_T)] + X_0 Y_0$$

holds for any admissible self-financing H and any local martingale deflator Y, and that there is equality if $U'(X_T) = Y_T$ and XY is a true martingale.

(d) Now consider the case where $n = 2$ and $P = (B, S)$, where

$$\begin{aligned} dB_t &= B_t r \; dt \\ dS_t &= S_t(\mu \; dt + \sigma \; dW_t) \end{aligned}$$

where r, μ, σ, B_0, S_0 are real constants such that $\sigma, B_0, S_0 > 0$ and where W is a Brownian motion. Show that there is a local martingale deflator Y with dynamics of the form

$$dY_t = -Y_t(r \; dt + \lambda \; dW_t)$$

where λ is a constant to be expressed in terms of the constants r, μ and σ. Show that if $H = (\phi, \pi)$ is a self-financing strategy and $X = \phi B + \pi S$ then

$$d(X_t Y_t) = Y_t(\pi_t S_t \sigma - X_t \lambda) dW_t$$

(e) Reconsider the utility maximisation problem of part (c) in the case where $U(x) = \log x$ and $\hat{U}(y) = -\log y - 1$ in the context of the model in part (d). Show that

$$\mathbb{E}(\log X_T) \leqslant \log X_0 + (r + \tfrac{1}{2}\lambda^2)T$$

with equality if $\pi_t = \frac{X_t \lambda}{S_t \sigma}$ for all $0 \leqslant t \leqslant T$.

[You may use standard results from stochastic calculus as long as they are stated clearly. You may assume all integrands are suitably integrable so that the stochastic integrals are well-defined.]

6 Let

$$dX_t = -\frac{1}{2}Z_t^2 dt + Z_t dW_t^X$$
$$dZ_t = B(Z_t)dt + C(Z_t)dW_t^Z$$

where W^X and W^Z are Brownian motions with constant correlation ρ. Fix a time horizon $T > 0$ and a function g, and let the smooth function U on $[0, T] \times \mathbb{R} \times \mathbb{R}$ solve the PDE

$$\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial z} + \frac{1}{2}C^2\frac{\partial^2 U}{\partial z^2} + zC\rho\frac{\partial^2 U}{\partial z \partial x} + \frac{1}{2}z^2\left(\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x}\right) = 0$$

with terminal condition

$$U(T, z, x) = g(x) \text{ for all } z, x \in \mathbb{R}.$$

(a) Show that the process $(M_t)_{0 \leqslant t \leqslant T}$ defined by $M_t = U(t, Z_t, X_t)$ is a local martingale.

Now suppose that $g(x) = e^{\theta x}$ for a constant θ.

(b) By making the substitution $U(t, z, x) = e^{\theta x}V(t, z)$ derive a PDE for V. What is the terminal condition?

Specialise to the case where $B(z) = a - bz$ and $C(z) = c$ for some constants a, b and c.

(c) Show there is a solution to the PDE derived in part (b) of the form

$$V(t, z) = e^{P(T-t)+Q(T-t)z+R(T-t)z^2}$$

for function P, Q and R satisfying a system of ordinary differential equations

$$\dot{R} = F(R)$$
$$\dot{Q} = G(Q, R)$$
$$\dot{P} = H(Q, R)$$

where $\dot{R}$ denotes the derivative of R, etc., and the functions F, G and H should be given explicitly in terms of the parameters a, b, c, ρ and θ.

[You may use standard facts from stochastic calculus without proof, as long as they are clearly stated.]

END OF PAPER