UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2022 9:00 am to 11:00 am

PAPER 204

PERCOLATION AND RELATED TOPICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be an infinite, locally-finite graph connected graph. Given G, let $\Omega_G = \{0,1\}^{E(G)}$.

(a) Define the terms *cylinder event* and *increasing cylinder event.*

(b) Let $V_1 \subset V_2 \subset \ldots \subset \bigcup_{n\geqslant 1} V_n = V(G)$ be an increasing sequence of subsets of $V(G)$, inducing connected subgraphs $G_1, G_2, \ldots$. Define the *free spanning forest* (FSF) μ^F induced on G by this exhaustion, in terms of measures of increasing cylinder events.

[You should state clearly any negative association results you use.]

(c) Prove that the FSF has no finite components almost surely.

(d) Let T be an infinite tree. Such a tree T is transient if and only if there exists an edge $e \in E(T)$ such that both components of $T \setminus e$ are transient.

Prove that if T is transient, FSF and WSF on T are not equal in distribution.

[You may use Wilson's algorithm rooted at infinity without defining it.]

2 For $d \geqslant 2$, consider the d-dimensional integer lattice $\mathbb{Z}^d$, and set $\Omega := \{0,1\}^{E(\mathbb{Z}^d)}$. Let $\mathbb{P}_p$ be the percolation product measure on Ω, with edge probability p.

(a) Let A be an event invariant under automorphisms of $\mathbb{Z}^d$.

Prove that $\mathbb{P}_p(A) = 0$ or 1.

[You may use the fact that $\forall \epsilon > 0$, there exists an event B that depends on only finitely many edges, such that $\mathbb{P}_p(A \triangle B) < \epsilon$, where the symmetric difference $A\triangle B$ is defined as $(A \setminus B) \cup (B \setminus A)$. You may also use the result that

$$\Big(A \cap \varphi(A)\Big) \triangle \Big(B \cap \varphi(B)\Big) \subset \Big(A \triangle B\Big) \cup \Big(\varphi(A) \triangle \varphi(B)\Big),$$

holds for $A, B \subset \Omega$, and φ an automorphism of $\mathbb{Z}^d$.]

(b) Let $N_\infty \in \{0, 1, \ldots, \} \cup \{\infty\}$ be the number of infinite open clusters in a configuration. It is known for every $p \in (p_c, 1]$, we have $\mathbb{P}_p(N_\infty \geqslant 1) = 1$. Prove that for every $p \in (p_c, 1]$, one of the following two statements is true:

$$\mathbb{P}_p(N_\infty = 1) = 1, \quad \mathbb{P}_p(N_\infty = \infty) = 1.$$

(c) For $k \geqslant 2$, let T_k be the rooted tree in which the root ρ has degree k, and every other vertex has degree $k+1$. Now let $\mathbb{P}_p$ denote percolation product measure on the edges of T_k. Show that for every $p \in [0, 1)$, one of the following two statements is true:

$$\mathbb{P}_p(N_\infty = 0) = 1, \quad \mathbb{P}_p(N_\infty = \infty) = 1.$$

3 Let $A \circ B$ denote the disjoint occurrence of two events $A, B \in \Omega_G$.

(a) State the *BK inequality* for the percolation product measure $\mathbb{P}_p$, including any conditions on the events.

(b) Prove, carefully, that on the d-dimensional integer lattice $\mathbb{Z}^d$, for $d \geqslant 2$,

$$\mathbb{P}_p\left(\exists \text{ two open edge-disjoint paths } 0 \leftrightarrow \infty\right) \leqslant \theta(p)^2,$$

where $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$ is the percolation probability.

(c) From now on, assume G is finite. Define the *random cluster measure* $\mathbb{P}_{\mathrm{FK}(p,q)}$ on Ω_G, with no boundary conditions.

(d) Let G be the cycle on four vertices, labelled (N, E, S, W) in that order. Consider the limit $p \to 1$, $q(1-p) \to \infty$. Show that

$$\mathbb{P}_{\mathrm{FK}(p,q)}(N \leftrightarrow S) = (1 + o(1))\frac{2(1-p)^2 q + 1}{(1-p)^4 q^3 + 1}.$$

(e) Hence, or otherwise, prove the existence of $p \in (0,1)$ and $q \geqslant 1$ and an event $A \subset \Omega_G$ such that

$$\mathbb{P}_{\mathrm{FK}(p,q)}(A) < 0.001 \quad \text{and} \quad \mathbb{P}_{\mathrm{FK}(p,q)}\left(A \circ A \mid A\right) > 0.999.$$

4

The graph $G = \mathbb{Z}^2 \oplus \mathbb{Z}^2$ is constructed by taking two disjoint copies of $\mathbb{Z}^2$, and for each labelled vertex $v \in \mathbb{Z}^2$, adding an edge between the two vertices with this label.

(a) Denote by $\mathcal{X} = (X_0, X_1, \ldots)$ a simple random walk on G, with $X_0 = x_0$ fixed. Prove that $\mathcal{X}$ is recurrent.

[You may use the fact that random walk on $\mathbb{Z}^2$ is recurrent without proof.]

(b) State the Aldous–Broder algorithm for constructing a uniform spanning tree T on G from $\mathcal{X}$.

(c) Given T, let T_1 and T_2 be the subgraphs of T induced on the two copies of $\mathbb{Z}^2$ which make up the vertex set of G.

Prove that T_1 almost surely contains a connected component of size at least 2022.

(d) Prove that T_1 is almost surely not connected.

END OF PAPER