UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 9:00 am to 11:00 am

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a compact $\mathbb{H}$-hull A and its mapping-out function g_A.

(b) Fix $r > 0$, $x \in \mathbb{R}$, let A be a compact $\mathbb{H}$-hull and let g_A be its mapping-out function. Moreover, let $rA = \{rz : z \in A\}$ and $A+x = \{z+x : z \in A\}$. Express the mapping-out functions g_{rA} and g_{A+x} in terms of g_A, r and x.

(c) Prove or disprove: Let $S = \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) \leqslant 1\}$. There exists a function g_S, satisfying the properties of a mapping-out function, with the set S in place of a compact $\mathbb{H}$-hull. [*You may use without proof that every (bijective) conformal map f from $\mathbb{H}$ onto $\mathbb{H}$ is of the form $f(z) = (az+b)/(cz+d)$ where $a, b, c, d \in \mathbb{R}$ are such that $ad - bc = 1$.*]

(d) Suppose that A is a compact $\mathbb{H}$-hull and let g_A be its mapping-out function. Let (z_n) be a sequence of points in $\mathbb{H} \setminus A$ such that $z_n \to z$ for some point $z \in A$. Show that $\mathrm{Im}(g_A(z_n)) \to 0$. Show by giving an example that it is not always true that $g_A(z_n)$ converges.

2

(a) State in full detail (including the random time-change) what it means for a Brownian motion to be conformally invariant.

(b) Let D be a simply connected domain. Does there exist a (bijective) conformal map $\varphi : D \to \mathbb{C} \setminus \{0\}$? Justify your answer.

(c) Let $\mathbb{P}^z$ denote the law of a complex Brownian motion B, with $B_0 = z$. For a compact $\mathbb{H}$-hull $A \subset \overline{\mathbb{D}}$, let g_A be its mapping-out function and let $\tau_A = \inf\{t \geqslant 0 : B_t \notin \mathbb{H} \setminus A\}$. Prove that for any Borel set $E \subset \mathbb{R} \setminus [-1, 1]$,

$$\lim_{y \to \infty} y \mathbb{P}^{iy}(B_{\tau_A} \in E) = \frac{\mathrm{Leb}(g_A(E))}{\pi},$$

where Leb denotes the one-dimensional Lebesgue measure. [*You may use without proof that if $\tau = \inf\{t \geqslant 0 : B_t \notin \mathbb{H}\}$ then for any Borel set $E \subset \mathbb{R}$,*

$$\mathbb{P}^{x+iy}(B_\tau \in E) = \int_E \frac{y}{(t-x)^2 + y^2} \frac{dt}{\pi}.$$

You may also use standard properties of g_A without proof, provided that you state them clearly.]

(d) Let $\tau_{\mathbb{D}} = \inf\{t \geqslant 0 : B_t \in \overline{\mathbb{D}}\}$. For each Borel set $E \subset \partial\mathbb{D}$, compute $\lim_{y \to \infty} \mathbb{P}^{iy}(B_{\tau_{\mathbb{D}}} \in E)$.

3

(a) State the Loewner differential equation for a given (continuous) driving function U and define the family of compact $\mathbb{H}$-hulls which is generated by U.

(b) Assume that $(K_t)_{t \geqslant 0}$ is the family of compact $\mathbb{H}$-hulls corresponding to an SLE process. State the conformal Markov property of (K_t).

(c) Prove that if U is the driving function of (K_t) and (K_t) satisfies the conformal Markov property then there exists a $\kappa > 0$ and a standard Brownian motion such that $U_t = \sqrt{\kappa} B_t$.

(d) Let (K_t) be the family of compact $\mathbb{H}$-hulls generated by some continuous, real-valued driving function U and let (g_t) denote the Loewner chain. Finally, let $T_x = \inf\{t \geqslant 0 : x \in K_t\}$. Prove using the Loewner differential equation, that if x is real and positive and $t < T_x$ then $g_t'(x) \in [0,1]$ (where $g_t'(z)$ denotes the derivative of $g_t(z)$ in the variable z). Prove also that $g_t'(x)$ is decreasing in t for $t < T_x$. [*You may without justification change the order of derivatives and you need not explain why* g_t' *extends continuously to* $\mathbb{R}$.]

4

(a) Let Λ be the set of values of κ for which an SLE_κ process intersects the boundary of the domain outside of the initial and terminal points. Describe Λ explicitly. That is, state which values of κ it contains.

(b) Prove that SLE_κ almost surely intersects the boundary outside of the initial and terminal points for $\kappa \in \Lambda$. [*You may use any properties of Bessel processes proved during the lectures, provided that you state them clearly.*]

(c) Explain how this can be used to deduce that SLE_κ is self-intersecting for $\kappa \in \Lambda$.

(d) Prove or disprove: Let $\eta \sim \mathrm{SLE}_\kappa$ for $\kappa \in \Lambda$ and let $\tau = \inf\{t \geqslant 0 : \eta(t) \notin \mathbb{D}\}$. Then $\mathbb{P}(\eta([0,\tau]) \cap \mathbb{R} = \{0\}) > 0$, that is, with positive probability η does not hit the boundary outside of its starting point before exiting $\mathbb{D}$.

END OF PAPER