UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 1:30 pm to 4:30 pm

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that if M is a continuous L^2-bounded martingale, then the norms

$$\|M\| := \|M_\infty\|_{L^2} \quad \text{and} \quad |||M||| := \left\| \sup_{t\geqslant 0} |M_t| \right\|_{L^2}$$

are equivalent.

(b) Let H be a simple process and M be a continuous L^2-bounded martingale. Define the Itô integral $H \cdot M$. Show that $H \cdot M$ is an L^2-bounded martingale.

(c) Let $\mathcal{P}$ be the previsible σ-algebra and ν be a finite measure on $\mathcal{P}$. Show that the set $\mathcal{S}$ of simple processes is dense in $L^2(\mathcal{P}, \nu)$.

(d) State the *Itô isometry*, carefully defining all the spaces involved. For M a continuous L^2-bounded martingale and H a simple process, prove the Itô isometry for $H \cdot M$.

[You may assume standard results involving $[M]$ when M is a continuous L^2-bounded martingale.]

2

(a) Define a previsible process. Give an example of a previsible process which is almost surely not left-continuous. Justify your answer in one line (no proof required).

(b) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and let Z be a uniformly integrable martingale with $Z_t > 0$ for all t. Define a new probability measure $\tilde{\mathbb{P}} \ll \mathbb{P}$ by setting $\tilde{\mathbb{P}}(A) = \mathbb{E}(Z_\infty \mathbf{1}_A)$ for all $A \in \mathcal{F}$. If Y is a uniformly bounded process, adapted to $(\mathcal{F}_t)$ and ZY is a continuous local martingale under $\mathbb{P}$, show that Y is a true martingale under $\tilde{\mathbb{P}}$.

(c) Let $(W, \mathcal{W}, \mathbb{P})$ be the Wiener space, i.e. $W = C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{W} = \sigma(X_t : t \geqslant 0)$, where $X_t \colon W \to \mathbb{R}$ is given by $X_t(w) = w(t)$, and $\mathbb{P}$ is the Wiener measure, i.e. the unique probability measure on $(W, \mathcal{W})$ such that $(X_t)_{t \geqslant 0}$ is a standard Brownian motion starting from 0. In this setting, state and prove the Cameron-Martin theorem.

[You may use the Girsanov Theorem. You may also use standard results like the Kunita-Watanabe identity, Levy's characterisation of Brownian motion, results on stochastic exponentials of martingales, quadratic variations under change of measures etc without proof.]

(d) Let B be a standard Brownian motion and for $a, b > 0$, let

$$\tau_{a,b} = \inf\{t \geqslant 0 : B_t + bt = a\}.$$

Show that the density of $\tau_{a,b}$ is given by

$$a(2\pi t^3)^{-1/2} \exp(-(a - bt)^2/(2t)).$$

[You may assume that the density for $\tau_{a,0}$ is $a(2\pi t^3)^{-1/2} \exp(-a^2/(2t))$.]

[You may use any standard results of martingale theory if you state them clearly. In particular, you can assume the result that a continuous local martingale which is locally in Doob's class is a true martingale.]

3

(a) Let $(M_t)_{t \geqslant 0}$ be a continuous local martingale with $M_0 = 0$.

1. Show that if $\mathbb{E}([M]_t) < \infty$ for all $t \geqslant 0$, then **both** M and $M^2 - [M]$ are true martingales.
2. Show that if $\mathbb{E}([M]_\infty) < \infty$, then M is an L^2-bounded martingale.

(b) Let B be a standard one-dimensional Brownian motion. Find

$$\mathbb{E}\left(B_t \int_0^t e^{B_s} dB_s \right).$$

[You may use the fact that the moment generating function $\mathbb{E}e^{\theta Z}$ of a standard Gaussian random variable Z is $e^{\frac{\theta^2}{2}}$.]

(c) Let $f \in C_b^2(\mathbb{R}), V \in C_b(\mathbb{R}), a \in C_b^1(\mathbb{R}), b \in C_b^1(\mathbb{R})$ with $a(x) \geqslant \epsilon$ for some $\epsilon > 0$ for all x. For any $g \in C_b^2(\mathbb{R})$, let

$$Lg(x) = b(x)g'(x) + \frac{1}{2}a(x)g''(x).$$

Let $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ solve

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = Lu(t,x) + V(x)u(t,x) \\ u(0,x) = f(x) \end{cases}$$

for all $t \in \mathbb{R}_+, x \in \mathbb{R}$. In this setting, state and prove the Feynman-Kac formula for X, where X is a solution to a suitable SDE to be specified.

[You may use any standard results proved in class or any results of martingale theory if you state them clearly (unless specifically asked to prove such a result). In particular, you can assume the result that a continuous local martingale which is locally in Doob's class is a true martingale. If you are using the Itô's formula, explicitly state your function and the vector of semimartingales.]

4

(a) Let B be a standard one-dimensional Brownian motion. Define

$$X_t = \int_0^t B_s^2 ds - B_t^2, \qquad t \geqslant 0\,.$$

Is X a local martingale? Justify your answer by clearly stating any standard result that you are using.

(b) Suppose that X is a continuous local martingale with quadratic variation

$$[X]_t = \int_0^t A_s ds$$

for a non-negative, previsible process $(A_t)_{t\geqslant 0}$. Show that there exists a Brownian motion B (possibly defined on a larger probability space) such that

$$X_t = X_0 + \int_0^t A_s^{1/2} dB_s.$$

(c) Let $b, \sigma : \mathbb{R} \mapsto \mathbb{R}$ be bounded, continuous functions. Define what it means for X to be an L-diffusion with diffusivity σ^2 and drift b. Show that if X is such an L-diffusion, then there exists a one-dimensional standard Brownian motion B such that $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

[You may use any standard results proved in class or any results of martingale theory if you state them clearly (unless specifically asked to prove such a result). If you are using the Itô's formula, explicitly state your function and the vector of semimartingales.]

5

(a) Let M, N be continuous local martingales.

1. Show that almost surely for all $t \geqslant 0$

$$|[M,N]_t| \leqslant \sqrt{[M]_t}\sqrt{[N]_t}\,.$$

2. If $V(t)$ denotes the total variation of $[M,N]$ on $[0,t]$, show that almost surely for all $t \geqslant 0$,

$$V(t) \leqslant \sqrt{[M]_t}\sqrt{[N]_t}\,.$$

[You may use any standard results about total variations, quadratic variations and covariations discussed in class.]

(b) Show that any non-negative integrable local martingale is a supermartingale.

(c) Define what is meant by a strong solution and a weak solution to the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t\,,$$

where B is a standard one-dimensional Brownian motion.

(d) Solve the following SDE:

$$dX_t = \sigma X_t dB_t + \mu X_t dt\,, \quad X_0 = x_0,$$

where $\mu, x_0 \in \mathbb{R}, \sigma > 0$ and B is a standard one-dimensional Brownian motion. Is it a strong solution?

[Hint: You may want to consider the function e^{aB_t+ct} for some appropriate constants a, c.]

[If you are using the Itô's formula, explicitly state your function and the vector of semimartingales. You may use any standard results if you state them clearly.]

6

(a) Fix $t > 0$ and let $h\colon [0,t] \to \mathbb{R}$ be a measurable function which is square-integrable, and let B be a standard one-dimensional Brownian motion. Show that $H_t = \int_0^t h(s)dB_s$ follows a Gaussian distribution, and compute its mean and variance.

(b) Let $\lambda > 0$. Solve the SDE

$$dX_t = -\lambda X_t dt + dB_t, \quad X_0 = x \in \mathbb{R},$$

where B is a standard Brownian motion.

[Hint: Consider the function $e^{\lambda t} X_t$.]

(c) For any fixed $t > 0$, show that X_t has a Gaussian distribution. Identify its mean and variance.

(d) Show that $(X_t)_{t>0}$ is a Gaussian process, that is, for all $0 < t_1 < \ldots < t_n$, $(X_{t_1}, \ldots, X_{t_n})$ is jointly Gaussian.

(e) For any $0 < s < t$, compute

$$\mathrm{Cov}(X_t, X_s)\,.$$

(f) If $X_0 \sim N(0, \frac{1}{2\lambda})$ independent of B, find the distribution of X_t and $\mathrm{Cov}(X_t, X_s)$ for $0 < s < t$.

[You may use any standard results if you state them clearly. If you are using the Itô's formula, explicitly state your function and the vector of semimartingales. You may assume standard facts about Gaussian distributions.]

END OF PAPER