UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 9:00 am to 12:00 pm

PAPER 160

REPRESENTATION THEORY OF SYMMETRIC GROUPS

Before you begin please read these instructions carefully.

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

All representations on this exam are assumed to be finite-dimensional.
Unless otherwise stated, they are over the field $\mathbb{C}$.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, let $\mathbb{F}$ be an arbitrary field. Let n be a natural number and let λ be a partition of n. Let the λ-Specht module over $\mathbb{F}$ be denoted by $\mathcal{S}^\lambda$, and let $t \in \Delta^\lambda$ be any λ-tableau.

(a) (i) Show that $\mathcal{S}^\lambda$ is generated as an $\mathbb{F}S_n$–module by the polytabloid $e(t)$.

(ii) Let $\{u\}$ be any λ-tabloid. Show that $\langle e(t), \{u\}\rangle \in \{0, \pm 1\}$.

(iii) Define a total ordering on the set of λ-tabloids, and use it to show that the polytabloids corresponding to standard λ-tableaux are linearly independent.

(b) Let $\mathfrak{b}_t$ denote the column symmetrizer of t. You may assume that $\mathfrak{b}_t \cdot M^\lambda = \mathbb{F}e(t)$. Show that if $s \in \Delta^\lambda$, then $\mathfrak{b}_t \cdot e(s) = \langle e(s), e(t)\rangle e(t)$.

Now for each $j \in [n]$, suppose that λ has a_j parts equal to j for some $a_j \in \mathbb{N}_0$. In other words, $\lambda = (n^{a_n}, \ldots, 2^{a_2}, 1^{a_1})$.

(c) Let $t^* \in \Delta^\lambda$ be obtained from t by reversing each row. For example, if

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}\ , \qquad \text{then} \quad t^* = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 \\ \cline{1-2} 7 & 6 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}\ .$$

(i) Suppose $h \cdot \{t\} = h^* \cdot \{t^*\}$ for some $h \in C(t)$ and $h^* \in C(t^*)$. Show that $h = h^*$.

[Hint: first consider $h(i)$ and $h^*(i)$ for i in the leftmost column of t.]

(ii) Deduce that if $\{u\}$ is a λ-tabloid such that $\langle e(t), \{u\}\rangle \neq 0$ and $\langle e(t^*), \{u\}\rangle \neq 0$, then $\{u\} = k \cdot \{t\}$ for some $k \in C(t) \cap C(t^*)$ and $\langle e(t), \{u\}\rangle = \langle e(t^*), \{u\}\rangle$.

(iii) Show that $\langle e(t), e(t^*)\rangle = \prod_{j=1}^{n} (a_j!)^j$.

(d) Now suppose $\mathrm{char}(\mathbb{F}) = p > 0$. Hence, or otherwise, show that

$$\dim_{\mathbb{F}} \left(\mathrm{End}_{\mathbb{F}S_n}(\mathcal{S}^\lambda) \right) = 1$$

whenever λ is p-regular.

2 For a partition λ, let χ^λ denote the character of the irreducible λ-Specht module over $\mathbb{C}$. In the usual notation from lectures, $\psi^\lambda = \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \cdot \xi^{\lambda - \mathrm{id} + \pi}$ for integer compositions λ, where $\xi^\lambda = \mathbb{1}_{S_\lambda}\big\uparrow^{S_n}$ if λ is a composition and $\xi^\lambda = 0$ otherwise.

(a) Let $n \in \mathbb{N}$ and suppose $n = m + k$ where $m, k \in \mathbb{N}_0$. Let λ be an integer composition of n. Prove that

$$\xi^\lambda \big\downarrow_{S_m \times S_k} = \sum_{\mu \vDash k} \xi^{\lambda - \mu} \# \xi^\mu .$$

Hence deduce that

$$\psi^\lambda \big\downarrow_{S_m \times S_k} = \sum_{\mu \vDash k} \psi^{\lambda - \mu} \# \xi^\mu .$$

[You may use earlier results from the course without proof, provided they are stated clearly.]

For $n \in \mathbb{N}$ and $\lambda, \mu \vdash n$, let $\chi^\lambda(\mu)$ denote the value of χ^λ on an element of S_n of cycle type μ. If λ and μ are both the empty partition, $\chi^\lambda(\mu) = 1$.

(b) (i) State the Murnaghan–Nakayama Rule.

(ii) If $\delta = (m, m-1, \ldots, 2, 1)$ for some $m \in \mathbb{N}$ and μ is a partition with $|\mu| = |\delta|$, show that $\chi^\delta(\mu) = 0$ whenever μ has a non-zero part of even size.

(iii) Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Prove that $\chi^{\lambda'} = \chi^\lambda \cdot \operatorname{sgn}_{S_n}$.

(c) Suppose $\lambda \vdash n$ has the property that $\chi^\lambda(\mu) = 0$ whenever $\mu \vdash n$ has a non-zero part of even size.

(i) Show that $\lambda = \lambda'$.

(ii) Deduce that either λ has no hooks of even size, or that the maximum even hook length of λ is attained by exactly two hooks of λ, namely as $h_{1,j}(\lambda)$ and $h_{j,1}(\lambda)$ for some $1 < j \leqslant \lambda_1$.

(iii) Hence, or otherwise, show that $\lambda = (m, m-1, \ldots, 2, 1)$ for some $m \in \mathbb{N}$.

3 Let $e \in \mathbb{N}$. Let λ be an arbitrary partition.

(a) (i) Prove that $|\lambda| = |C_e(\lambda)| + e\mathbf{w}_e(\lambda)$, where $C_e(\lambda)$ denotes the e-core of λ and $\mathbf{w}_e(\lambda)$ the e-weight of λ.

[You may use earlier results from the course without proof, provided they are stated clearly.]

(ii) Determine with proof when the e-quotient tower $T^Q(\lambda)$ of λ has finite depth.

(b) Suppose that the e-quotient of λ is $Q_e(\lambda) = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(e-1)}\right)$. Describe how to calculate $Q_e(\lambda)$ using James's e-abacus. Prove that

$$Q_e(\lambda') = \left((\lambda^{(e-1)})', \ldots, (\lambda^{(1)})', (\lambda^{(0)})'\right).$$

[You may use results from example sheets without proof, provided they are stated clearly.]

(c) Let $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We define the e-residue of (i,j) to be the value $r_e(i,j) \in \{0, 1, \ldots, e-1\}$ such that $r_e(i,j) \equiv j - i \pmod{e}$. The e-content of λ is defined to be the multiset $\{r_e(i,j) \mid (i,j) \in \mathcal{Y}(\lambda)\}$.

For example, if $\alpha = (5,4,4,2,1) \vdash 16$ then $\mathcal{Y}(\alpha)$ with $r_4(i,j)$ filled into each box (i,j) is given by

0	1	2	3	0
3	0	1	2	
2	3	0	1	
1	2			
0				

and the 4-content of α is $\{0,0,0,0,0,1,1,1,1,2,2,2,2,3,3,3\}$.

(i) Suppose λ and μ are two partitions of the same size. Show that if λ and μ have equal e-cores, then they have equal e-contents.

(ii) Let $m \in \mathbb{N}$ be a multiple of e, and suppose A is the e-abacus configuration of a β-set $\mathbf{X} = \{h_1, h_2, \ldots, h_m\}$ where $h_1 > h_2 > \cdots > h_m \geqslant 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be the partition corresponding to A (where this expression may contain trailing zeros). Show that for each $i \in \{1, 2, \ldots, m\}$,

$$r_e(i, \lambda_i) \equiv h_i \pmod{e}.$$

(iii) Suppose λ and μ are two partitions of the same size. Show that if λ and μ have equal e-contents, then they have equal e-cores.

4 Let $\mathbb{F}$ be a field. For a partition μ, the μ-Young permutation module over $\mathbb{F}$ is denoted by M^{μ}, and the μ-Specht module over $\mathbb{F}$ by $\mathcal{S}^{\mu}$.

(a) Recall that $G(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ is the generating function counting the number of partitions (i.e. $G(x) = \sum_{n=0}^{\infty} |\mathcal{P}(n)| x^n$).

(i) Let $e \in \mathbb{N}$. Write down the generating function counting the number of partitions into parts of size at most e. Explain why it is equal to the generating function counting the number of partitions into at most e parts.

(ii) Let $e \in \mathbb{N}$. Write down the generating function counting the number of partitions into exactly e parts.

(iii) Let $n \in \mathbb{N}$. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts. Hence write down the generating function counting the number of self-conjugate partitions.

(b) Suppose $\mathbb{F} = \mathbb{C}$. Let $n \in \mathbb{N}$ and let λ be a partition of n. Suppose we have an isomorphism of $\mathbb{C}S_n$–modules $M^{\lambda} \cong \bigoplus_{\alpha \vdash n} (\mathcal{S}^{\alpha})^{\oplus m_\alpha}$ for some $m_\alpha \in \mathbb{N}_0$.

(i) If $\alpha \vdash n$ is such that $m_\alpha > 0$, prove that $\alpha \trianglerighteq \lambda$.
[You may assume that if $t \in \Delta^{\alpha}$ and $u \in \Delta^{\lambda}$ are such that $\mathfrak{b}_t \cdot \{u\} \neq 0$, then $\alpha \trianglerighteq \lambda$.]

(ii) For any $\alpha \vdash n$, describe the value of m_α in terms of the number of certain tableaux.

(iii) Let $n \geqslant 3$ and $\lambda = (n-2, 1^2)$. Determine m_α for all $\alpha \vdash n$.

(c) Now let $\mathbb{F}$ be any field. Let $n \in \mathbb{N}$ with $n \geqslant 3$. Construct, with proof, a sequence of submodules $U_0, U_1, \ldots, U_k$ of $M^{(n-2,1^2)}$ (for some k) with the following properties:

- $U_0 = 0$ and $U_k = M^{(n-2,1^2)}$;
- $U_0 < U_1 < U_2 < \cdots < U_{k-1} < U_k$; and
- for each $i \in \{0, 1, \ldots, k-1\}$, U_{i+1}/U_i is isomorphic as $\mathbb{F}S_n$–modules to $\mathcal{S}^{\alpha(i)}$ for some partition $\alpha(i) \vdash n$. The partitions $\alpha(i)$ should be explicitly determined.

[Hint: when $n \geqslant 4$, first show that

$$0 \leqslant \mathcal{S}^{(n-2,1^2)} \leqslant V \cap \ker \phi_2 \leqslant V \leqslant U \cap \ker \phi_1 \leqslant U \leqslant \ker \phi_0 \leqslant M^{(n-2,1^2)}$$

for suitably defined S_n–homomorphisms $\phi_i : M^{(n-2,1^2)} \to M^{(n-i,i)}$ for $i \in \{0, 1, 2\}$, and suitably defined $U, V \leqslant M^{(n-2,1^2)}$.]

END OF PAPER