UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2022 9:00 am to 11:00 am

PAPER 156

MAPPING CLASS GROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let S_g be the closed, orientable surface of genus g.

(a) Let $g \geqslant 2$, and suppose that S_g is equipped with a hyperbolic metric. Prove that any finite group G of orientation-preserving isometries of S_g embeds into the mapping class group $\mathrm{Mod}(S_g)$. Give examples to show that the analogous statements are false when $g = 0$ and $g = 1$.

(b) Show that for any finite group G there is a closed surface S such that G is a subgroup of $\mathrm{Mod}(S)$.

(c) If a finite group G acts freely by orientation-preserving isometries on a hyperbolic surface S_g, prove that $|G| \leqslant g-1$. [You may use standard properties of Euler characteristic without proof.]

(d) Give an example of an integer $g > 1$ and a finite group G acting by orientation-preserving isometries on the hyperbolic surface S_g such that $|G| > g - 1$.

2 Let S be a non-compact surface of finite type, endowed with a complete hyperbolic metric of finite area. Let α, β be a transverse pair of properly embedded, essential, simple arcs on S.

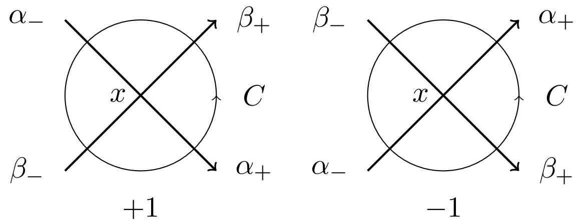
(a) What does it mean for α and β to *bound a bigon*? What does it mean for α and β to be in *minimal position*?

(b) Prove that, if α and β do not bound a bigon, then every pair of lifts of α and β to $\mathbb{H}^2$ intersect in at most one point of the compactified hyperbolic plane $\overline{\mathbb{H}}^2$.

[*Hint: You may use the following fact without proof. If $\phi \in \pi_1(S)$ acts as a hyperbolic isometry on $\mathbb{H}^2$ and $\psi \in \pi_1(S)$ acts as a parabolic isometry on $\mathbb{H}^2$ then the fixed point of ψ in $\partial\mathbb{H}^2$ is not fixed by ϕ.*]

(c) Explain how to adapt the proof of the bigon criterion for simple closed curves on S to simple proper arcs on S: if α and β are not isotopic, then they are in minimal position if and only if they do not bound a bigon.

3 Let S be a connected, oriented surface of finite type. Consider an ordered pair of transverse, oriented essential simple closed curves α, β on S. The orientations allow us to assign a sign $\sigma(x) = \pm 1$ to each point x at which α and β cross, as in the picture.



Precisely, let C be a small circle around x, with orientation induced by the orientation on S. Let $\alpha \cap C = \{\alpha_-, \alpha_+\}$, where α_- is before x and α_+ is after x on α, and let $\beta \cap C = \{\beta_-, \beta_+\}$ similarly. Then $\sigma(x) = +1$ if the 4-tuple $(\alpha_+, \beta_+, \alpha_-, \beta_-)$ appears anticlockwise in C, and $\sigma(x) = -1$ if $(\alpha_+, \beta_+, \alpha_-, \beta_-)$ appears clockwise in C.

The *algebraic intersection number* of α and β is defined to be

$$\langle \alpha, \beta \rangle := \sum_{x \in \alpha \cap \beta} \sigma(x)\,.$$

(a) Using suitable results from the course, prove that there are essential simple closed curves α_0, β_0, isotopic to α and β respectively, such that

$$\langle \alpha, \beta \rangle = \langle \alpha_0, \beta_0 \rangle$$

and α_0, β_0 are in minimal position.

(b) If $\alpha_0 \simeq \alpha_1$ with α_1, β_0 also in minimal position, prove that $\langle \alpha_0, \beta_0 \rangle = \langle \alpha_1, \beta_0 \rangle$, again using suitable results from the course.

(c) Show that $\langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle$.

(d) Give an example such that $\langle \alpha, \beta \rangle = 0$ but $i(\alpha, \beta) \neq 0$.

4 Let S be a non-compact, connected, oriented surface of finite type, with $\partial S = \varnothing$. Recall that a simple proper arc on S is *essential* unless it is isotopic (rel. endpoints) into a puncture.

(a) Give a definition of an *arc complex* $A(S)$, in analogy with the curve graph $C(S)$, where the vertices are isotopy classes of essential, unoriented, simple proper arcs embedded in S, equipped with an action of $\mathrm{Mod}(S)$ on $A(S)$.

(b) Let $S = S_{0,3,0}$, the three-punctured sphere. For a given puncture p of S, classify the isotopy classes of essential arcs starting and ending at p. Give a complete description of $A(S)$ and the action of $\mathrm{Mod}(S)$ on $A(S)$.

(c) Now consider the case when $S = S_{0,4,0}$, the four-punctured sphere, and let p be a puncture of S. Prove that there are infinitely many isotopy classes of simple proper arcs starting and ending at p. How many $\mathrm{Mod}(S)$-orbits of vertices are there in $A(S)$?

[*You may use suitable adaptations of results about simple closed curves to simple proper arcs without proof, as long as you state them clearly.*]

END OF PAPER