UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 1:30 pm to 4:30 pm

PAPER 154

INTRODUCTION TO NON-LINEAR ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Bound state for a modified harmonic oscillator

We work in dimension $d \geqslant 1$ with real valued functions. For $x \in \mathbb{R}^d$, we let $|x| = \sqrt{\sum_{i=1}^d x_i^2}$. We pick $\alpha > 0$ and consider the space

$$\Sigma = H^1 \cap \{|x|^{\frac{\alpha}{2}} u \in L^2\}$$

equipped with the scalar product

$$\langle u, v \rangle_\Sigma = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v dx + \int_{\mathbb{R}^d} |x|^\alpha uv dx.$$

1. Show that there exists a universal constant $C > 0$ such that

$$\forall u \in \mathcal{D}(\mathbb{R}^d), \quad \int_{|x| \leqslant 1} |u|^2 \leqslant C \|u\|_\Sigma^2.$$

   Conclude that $(\Sigma, \langle \cdot, \cdot \rangle_\Sigma)$ is a Hilbert space.
   (*Hint: integrate parts using that for any well localized function* χ *which is one on* $|x| \leqslant 1$, $\nabla \cdot (x\chi) = d\chi + x \cdot \nabla \chi$)

2. Prove that the embedding $\Sigma \subset L^2$ is continuous and compact.

3. Show that for all $f \in L^2$, $\exists! T(f) \in \Sigma$ such that

$$\forall v \in \Sigma, \quad \langle v, T(f) \rangle_\Sigma = \langle v, f \rangle_{L^2}$$

   and that the map $T : L^2 \mapsto \Sigma$ is continuous.

4. Show that the map $T : L^2 \mapsto L^2$ is compact and derive the equation satisfied by $T(f)$ in $\mathcal{D}'(\mathbb{R}^d)$.

5. Show that there exists $\lambda > 0$ and $\psi \geqslant 0$, $\psi \in \Sigma$ non zero, solution to

$$-\Delta + |x|^\alpha \psi = \lambda \psi \;\text{ in }\; \mathcal{D}'(\mathbb{R}^d).$$

2 Small data global existence for critical NLS

We work in dimension $d \geqslant 1$ with complex valued functions. We let $u \in H^1_{\text{rad}}$ be the space of $H^1(\mathbb{R}^d)$ distributions with radial symmetry. We define the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{2+\frac{4}{d}}\int_{\mathbb{R}^d} |u|^{2+\frac{4}{d}} dx$$

and for $u \in H^1$ non zero,

$$J(u) = \frac{\|\nabla u\|^2_{L^2}\|u\|^{\frac{4}{d}}_{L^2}}{\|u\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}}}.$$

We let Q be the radially symmetric ground state solution to $\Delta Q - Q + Q^{2+\frac{4}{d}} = 0$.

1. Given $a \in \mathbb{C}$, $\lambda > 0$, let $u_{a,\lambda}(x) = au(\lambda x)$. Show that $J(u_{a,\lambda}) = J(u)$. Show that

$$I = \inf_{u \in H^1 \backslash \{0\}} J(u) > 0.$$

2. Show that the infimum is attained and that there exists a minimizer which is non negative. You may assume that

$$I = \inf_{u \in H^1_{\text{rad}} \backslash \{0\}} J(u).$$

3. Show that a minimizer $u \geqslant 0$ satisfies an equation of the form

$$\Delta u - \lambda u + \mu u^{1+\frac{4}{d}} = 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^d)$$

for some $\lambda = \lambda_u > 0, \mu = \mu_u > 0$. Classify all minimizers.

4. Show that

$$I = \frac{2}{2+\frac{4}{d}}\|Q\|^{\frac{4}{d}}_{L^2}.$$

hint: Show that $E(Q) = 0$ using the Pohozaev multiplier (no need to reprove the Pohozaev identity).

5. Prove that

$$\forall u \in H^1, \quad E(u) \geqslant \frac{1}{2}\|\nabla u\|^2_{L^2}\left[1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{d}}\right].$$

3 Minimal mass blow up solutions

We work in dimension $d \geqslant 1$ with complex valued functions. We let $u \in H^1_{\text{rad}}$ be the space of $H^1(\mathbb{R}^d)$ distributions with radial symmetry. We define the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{2+\frac{4}{d}}\int_{\mathbb{R}^d} |u|^{2+\frac{4}{d}} dx.$$

We let Q be the radially symmetric ground state solution to $\Delta Q - Q + Q^{2+\frac{4}{d}} = 0$. We **admit** the following sharp Gagliardo-Nirenberg inequality :

$$\forall u \in H^1, \quad E(u) \geqslant \frac{1}{2}\|\nabla u\|_{L^2}^2 \left[1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{d}}\right].$$

1. Let $u_0 \in H^1$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, show that the unique corresponding solution to the (NLS) problem

$$(NLS) \quad \left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^{\frac{4}{d}} = 0 \\ u_{|t=0} = u_0 \end{array} \right.$$

   is global in time.

2. Given $h \in \mathcal{D}(\mathbb{R}^d)$ real valued, compute

$$\frac{d}{dt}E(Q+th)_{|t=0}.$$

   Conclude that $\forall \varepsilon > 0$, there exist an initial data $u_0 \in H^1$ with $\|u_0\|_{L^2} < \|Q\|_{L^2} + \varepsilon$ such that the corresponding solution to (NLS) blows up in finite time.

3. Let $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Assume that the corresponding solution $u(t,\cdot)$ to (NLS) blows up in finite time say $0 < T < +\infty$. Let $t_n \to T$. Compute $\lambda(t_n)$ so that

$$v_n(x) = \lambda_n^{\frac{d}{2}} u\left(t_n, \lambda_n x\right)$$

   satisfies

$$\forall n, \quad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

   Compute $\lim_{n\to+\infty} E(v_n)$.

4. State the profile decomposition Lemma.

5. Show that there exists $x_n \in \mathbb{R}^d$ such that up to a subsequence, $v_n(\cdot + x_n)$ is strongly convergent in $L^{2+\frac{4}{d}}$.

END OF PAPER