UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 9:00 am to 12:00 pm

## PAPER 152

## TORIC GEOMETRY

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

(a) Let $\sigma \subseteq \mathbb{R}^2$ be the cone with generators $(1,0)$ and $(-1,3)$.

(i) Express $\mathbb{C}[S_\sigma]$ as a subring of $\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. [You do not need to find the relations between the generators.]

(ii) Construct an isomorphism

$$X_\sigma \cong \mathbb{A}^2_{xy}/\mu_3$$

where the action $\mu_3 \curvearrowright \mathbb{A}^2_{xy}$ is given by

$$\zeta(x) = \zeta \cdot x, \quad \zeta(y) = \zeta \cdot y$$

where $\zeta \in \mu_3$ is a primitive 3rd root of unity.

(iii) Construct a toric resolution of singularities of $X_\sigma$. Calculate the self-intersection number(s) of the resulting exceptional curve(s).

(b) Let $\Sigma$ be a fan in $\mathbb{R}^2$ with $|\Sigma| = \mathbb{R}^2$ and consider a subdivision $\Sigma^\dagger \to \Sigma$. Assume that $X_{\Sigma^\dagger}$ is smooth and let $\tau \in \Sigma^\dagger(1) \setminus \Sigma(1)$, i.e. $\tau$ is a ray in $\Sigma^\dagger$ but not in $\Sigma$. Prove that

$$D_\tau^2 < 0.$$

(c) Recall that the $A_{m-1}$ singularity is the affine toric variety $X_\sigma$ associated to the cone $\sigma \subseteq \mathbb{R}^2$ with generators $(1,0)$ and $(1,m)$. Prove that for $m \geqslant 3$ there is no proper toric variety $X_\Sigma$ satisfying all of the following conditions simultaneously:

- $X_\Sigma$ contains $X_\sigma$ as the toric affine open corresponding to a cone $\sigma \in \Sigma$.
- $X_\Sigma$ is smooth everywhere except at the distinguished point $x_\sigma \in X_\sigma \subseteq X_\Sigma$.
- $\Sigma$ has precisely 3 rays.

**2** *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

(a) Fix strictly positive integers $a, b$ with $\gcd(a, b) = 1$. Consider the fan $\Sigma$ in $\mathbb{R}^2$ with $|\Sigma| = \mathbb{R}^2$ and with rays generated by $(1, 0), (0, 1), (-a, -b)$. The toric variety $X_\Sigma$ is known as the weighted projective space, and we write $X_\Sigma = \mathbb{P}(1, a, b)$.

(i) For which values of $a, b$ is $\mathbb{P}(1, a, b)$ smooth?

(ii) Calculate $\mathrm{Cl}\ \mathbb{P}(1, a, b)$.

(iii) Calculate $\mathrm{Pic}\ \mathbb{P}(1, a, b)$ and describe the homomorphism

$$\mathrm{Pic}\ \mathbb{P}(1, a, b) \to \mathrm{Cl}\ \mathbb{P}(1, a, b).$$

(iv) Apply the quotient description of toric varieties given in lectures to show that, at the level of closed points, $\mathbb{P}(1, a, b)$ coincides with the quotient

$$(\mathbb{A}^3_{xyz} \setminus \{0\})/\mathbb{C}^\star$$

where the action $\mathbb{C}^\star \curvearrowright \mathbb{A}^3_{xyz}$ is given by:

$$t(x) = t \cdot x, \quad t(y) = t^a \cdot y, \quad t(z) = t^b \cdot z.$$

(b) Using the fans, prove that there is no non-constant toric morphism $\mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$.

**3** *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

Recall that the Hirzebruch surface $\mathbb{F}_1$ is the toric surface associated to the fan $\Sigma$ with $|\Sigma| = \mathbb{R}^2$ and with rays generated by $(1,0), (0,1), (-1,1), (0,-1)$. Let $D_1 \subseteq \mathbb{F}_1$ be the toric hypersurface corresponding to the ray $(1,0)$.

(a) Calculate $\dim \Gamma(\mathbb{F}_1, \mathcal{O}_{\mathbb{F}_1}(kD_1))$ for all $k \in \mathbb{Z}$.

(b) Prove that $\mathcal{O}_{\mathbb{F}_1}(kD_1)$ is basepoint-free for all $k \geqslant 0$.

(c) Prove that $\mathcal{O}_{\mathbb{F}_1}(kD_1)$ is not ample for any $k \geqslant 0$.

Consider the morphism of lattices $\mathbb{Z}^2 \to \mathbb{Z}$ given by $(u_1, u_2) \mapsto u_1$. This is a morphism from the fan of $\mathbb{F}_1$ to the fan of $\mathbb{P}^1$ and hence induces a toric morphism $\mathbb{F}_1 \to \mathbb{P}^1$ which we denote by $\pi$.

(d) Find a toric Cartier divisor $E_1$ on $\mathbb{P}^1$ such that $D_1 = \pi^{-1}(E_1)$.

(e) Prove or disprove: every line bundle on $\mathbb{F}_1$ is isomorphic to $\pi^\star L$ for some line bundle $L$ on $\mathbb{P}^1$.

## END OF PAPER