UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 9:00 am to 12:00 pm

PAPER 144

MODEL THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **SIX** questions.
There are **EIGHT** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let T be a theory in a language L. Suppose that whenever $\mathcal{A} \hookrightarrow \mathcal{M} \models T$ is a substructure of a model of T then the theory $T \cup D(\mathcal{A})$ is complete. Show that T admits quantifier elimination in the language L.

2 Show that the theory of algebraically closed fields has quantifier elimination in the language of rings. Let $F \subseteq K$ be a subfield of an algebraically closed field. Describe in algebraic terms the Stone Spaces $S_n^K(F)$, identifying the complete types and the Stone Topology.

3 Consider RCF, the theory of real closed fields in the language of rings, and recall that $x \leqslant y$ if and only if $\exists z.z^2 = y - x$ defines an order.

(i) Show that RCF does not admit quantifier elimination in the language of rings.

(ii) Show that there exist non-Archimedean models of RCF, that is models containing α with $\alpha > n$ for all n.

(iii) Is RCF ω-categorical? Is it uncountably categorical? Briefly justify your answers.

(iv) RCF is model complete in the sense that if $F \subseteq K$ are real closed fields then $F \preceq K$. Explain briefly how this fact is used to show that any positive semi-definite function in $\mathbb{R}(X_1, \cdots, X_n)$ can be written as a sum of squares.

4 (i) What is an atomic model of a theory T? Suppose that $\mathcal{M}$ and $\mathcal{N}$ are infinite atomic models of a complete theory T. Show that given a partial elementary map $\mathbf{a} \to \mathbf{b}$ from $\mathcal{M}$ to $\mathcal{N}$, for any c in $\mathcal{M}$ we can find d in $\mathcal{N}$ such that $\mathbf{a}, c \to \mathbf{b}, d$ is partial elementary. Why does it follow that countable atomic models of a complete T are isomorphic?

(ii) Let T be complete with infinite models in a countable language. Show that T is ω-categorical if and only if all Stone Spaces $S_n(T)$ are finite.

5 What is the the theory RG of the random (or Rado) graph? Show that RG is consistent. Show that RG has quantifier elimination in the language of graphs. Describe the Stone Space $S_3(\mathrm{RG})$. How many elements does it have?

6 (i) Let κ be an infinite cardinal. What is a κ-saturated structure and what is a κ-homogeneous structure? Show that a κ-saturated structure is κ-homogeneous.

(ii) Suppose that $\mathcal{M}$ is an ω-homogeneous model of a complete theory T. Show that if all types in the Stone Spaces $S_n(T)$ are realised in $\mathcal{M}$ then $\mathcal{M}$ is ω-saturated.

7 Let T be a complete theory with infinite models in a countable language. Show that if T is ω-stable then T is κ-stable for all infinite cardinals κ.

8 (i) For λ an infinite cardinal consider $I = \mathbb{Q}^\lambda$ with the lexicographic ordering: $f < g$ if and only if $f(\alpha) < g(\alpha)$ for α least such that $f(\alpha) \neq g(\alpha)$. Show that the set J of eventually 0 elements is dense in I. If λ is least such that $2^\lambda > \kappa$ for some cardinal κ show that $|J| \leqslant \kappa$.

(ii) What does it mean for a formula $\phi(\mathbf{x}, \mathbf{y})$ to have the order property with respect to a theory T? Suppose that $\phi(\mathbf{x}, \mathbf{y})$ has the order property with respect to some complete T with infinite models. Show that T is not κ-stable for any κ greater than or equal to the cardinality of T.

END OF PAPER