UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 9:00 am to 11:00 am

PAPER 140

SYMPLECTIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 On $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2), consider the smooth family of 1-forms

$$\sigma_s := \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + \frac{s}{2}(x_2 dy_2 - y_2 dx_2),$$

where $s \in (0, \infty)$. Consider the unit 3-sphere $S^3 \subset \mathbb{R}^4$ given by $\sum_{i=1}^{2}(x_i^2 + y_i^2) = 1$ and let $\iota : S^3 \to \mathbb{R}^4$ denote the inclusion map. Show that for each $s \in (0, \infty)$, $\alpha_s := \iota^* \sigma_s$ defines a contact form on S^3. Find the Reeb vector field R_s associated to α_s. Prove that for s irrational R_s has only two closed orbits.

2 Let (M, ω) be a symplectic manifold. Suppose X is a compact Lagrangian submanifold with $H^1(X, \mathbb{R}) = \{0\}$. Show that every embedding $j : X \to M$ which is sufficiently C^1-close to the inclusion $X \hookrightarrow M$ and such that $j(X)$ is Lagrangian, has the property that $j(X) \cap X$ has at least two points. Is the claim still true if we drop the assumption that $H^1(X, \mathbb{R}) = \{0\}$?

[You may assume the Weinstein tubular neighbourhood theorem, provided it is clearly stated. You may also assume that a smooth map which is C^1-close to the identity is a diffeomorphism.]

3 (a) Let X be an n-dimensional manifold and let α denote the Liouville 1-form of T^*X. Given a diffeomorphism $f : X \to X$ explain how to lift it to a natural diffeomorphism $f_\# : T^*X \to T^*X$ such that $f_\#^* \alpha = \alpha$.

(b) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $\omega = -d\alpha$ the canonical symplectic form of T^*S^2 with projection $\pi : T^*S^2 \to S^2$. Consider the 2-form σ on S^2 given by

$$\sigma_u(v, w) = \langle u, v \times w \rangle,$$

where $u \in S^2$, $v, w \in T_u S^2$ are vectors in $\mathbb{R}^3$, $\times$ is the cross product and $\langle \cdot, \cdot \rangle$ is the standard inner product. Consider the 2-forms on T^*S^2

$$\omega_\pm = \omega \pm \pi^* \sigma.$$

Show that ω_+ and ω_- are symplectic forms. Are ω_+ and ω_- strongly isotopic? Are ω_+ and ω_- symplectomorphic?

4 Let L be a Lagrangian submanifold of a symplectic manifold (M, ω) and $H : M \to \mathbb{R}$ a Hamiltonian such that H restricted to L is constant. Show that the Hamiltonian flow of H preserves L.

Let V be a vector field on a manifold X with flow ϕ_t defined for all $t \in \mathbb{R}$. Consider T^*X with its standard symplectic form and for each $t \in \mathbb{R}$, let $\psi_t := (\phi_t)_\# : T^*X \to T^*X$ denote the lift of ϕ_t to T^*X. Show that ψ_t is also a flow, that is, $\psi_{t+s} = \psi_t \circ \psi_s$ for all $s, t \in \mathbb{R}$. Let $V_\#$ denote the vector field of the flow ψ_t, that is,

$$V_\#(x, \xi) := \left.\frac{d}{dt}\right|_{t=0} \psi_t(x, \xi), \quad \text{for } (x, \xi) \in T^*X$$

Show that $V_\#$ is the Hamiltonian vector field of the Hamiltonian $H : T^*X \to \mathbb{R}$ given by $H(x, \xi) = \xi(V(x))$.

Let $Y \subset X$ be a submanifold invariant under the flow ϕ_t. Show that the conormal bundle N^*Y of Y is invariant under the flow ψ_t.

END OF PAPER