UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 1:30 pm to 4:30 pm

PAPER 137

MODULAR FORMS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $k \in \mathbb{Z}$ and let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Define the space $M_k(\Gamma)$ of modular forms of weight k and level Γ.

(b) Now let $N \in \mathbb{N}$ and assume that $k > 2$. For each $(x, y) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $\tau \in \mathfrak{h}$, define

$$G_k^{(x,y)}(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2-\{(0,0)\} \\ (c,d)\equiv(x,y) \bmod N}} (c\tau + d)^{-k}.$$

Prove that $G_k^{(x,y)}(\tau)$ converges absolutely to a holomorphic function in $\mathfrak{h}$.

(c) Show that if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $G_k^{(x,y)}|_k[\gamma] = G_k^{(x,y)\gamma}$.

(d) Prove that $G_k^{(x,y)} \in M_k(\Gamma(N))$.

2 Let $k \geqslant 12$ be an even integer.

(a) Let

$$\mathcal{F} = \{\tau \in \mathfrak{h} \mid |\tau| \geqslant 1, \mathrm{Re}(\tau) \in [-1/2, 1/2]\}.$$

Show that every element of $\mathfrak{h}$ is $\mathrm{SL}_2(\mathbb{Z})$-conjugate to an element of $\mathcal{F}$.

(b) Show that if $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ has q-expansion $f(\tau) = \sum_{n=1}^{\infty} a_n(f)q^n$, then there is a constant $C > 0$ such that $|a_n(f)| \leqslant Cn^{k/2}$ for each $n \in \mathbb{N}$.

(c) Let $G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}}(c\tau + d)^{-k} \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that if $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, then the integral

$$\int_{\tau\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}} f(\tau)\overline{G_k(\tau)}y^k \frac{dxdy}{y^2}$$

(where $\tau = x + iy$) is absolutely convergent, and compute its value.

3 Let N be a positive integer, and let p be a prime number.

(a) Define the congruence subgroup $\Gamma_1(N)$.

(b) Let $\mathcal{L}(N)$ denote the set of pairs $(\Lambda, v + \Lambda)$, where $\Lambda \leqslant \mathbb{C}$ is a lattice and $v+\Lambda \in \mathbb{C}/\Lambda$ is a point of exact order N, in the sense that $Nv+\Lambda = 0+\Lambda$ but $dv+\Lambda \neq 0+\Lambda$ for any $d \in \mathbb{N}$, $1 \leqslant d < N$. Let $\mathbb{C}^\times$ act on $\mathcal{L}(N)$ by $\lambda(\Lambda, v + \Lambda) = (\lambda\Lambda, \lambda v + \lambda\Lambda)$.

Show that the map $\tau \mapsto (\Lambda_\tau, 1/N + \Lambda_\tau)$, where $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$, determines a bijection

$$\Gamma_1(N)\backslash\mathfrak{h} \xrightarrow{\sim} \mathbb{C}^\times\backslash\mathcal{L}(N).$$

(c) Let $(\Lambda, v + \Lambda) \in \mathcal{L}(N)$. Let $a_p(\Lambda, v + \Lambda)$ denote the number of lattices $\Lambda' \leqslant \mathbb{C}$ such that $\Lambda \leqslant \Lambda'$, $[\Lambda' : \Lambda] = p$, and the image $v + \Lambda'$ of $v + \Lambda$ in $\mathbb{C}/\Lambda'$ has exact order N.

Compute $a_p(\Lambda, v + \Lambda)$ when (i) p does not divide N and (ii) when p divides N.

4 Let $k \geqslant 2$ be an even integer.

(a) Let $\Lambda \leqslant \mathbb{R}^n$ be a lattice. Define the dual lattice $\Lambda^\vee \leqslant \mathbb{R}^n$. State and prove a version of the Poisson summation formula for a continuous function $f : \mathbb{R}^n \to \mathbb{C}$ and lattice Λ.

(b) Let us identify $\mathbb{C} = \mathbb{R}^2$ using the basis $1, i$. Given a lattice $\Lambda \leqslant \mathbb{C}$, define

$$\theta_k(\Lambda) = \sum_{\lambda \in \Lambda} \lambda^k e^{-\pi|\lambda|^2}.$$

Prove the identity

$$\theta_k(\Lambda) = (-i)^k m(\Lambda)^{-1} \theta_k(\Lambda^\vee).$$

[You may use the identity $\hat{f}_k(x,y) = (-i)^k f_k(x,y)$, where $\hat{f}_k(x,y)$ is the Fourier transform of the function $f_k : \mathbb{R}^2 \to \mathbb{C}$ given by the formula $f_k(x,y) = (x+iy)^k e^{-\pi(x^2+y^2)}$.]

(c) For $\tau \in \mathfrak{h}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(k+2s) > 2$, let

$$G_k(\tau, s) = \sum_{(c,d) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{\mathrm{Im}(\tau)^s}{(c\tau + d)^k |c\tau + d|^{2s}}$$

(you may assume that the sum converges absolutely for such values of s).

Prove that for each fixed $\tau \in \mathfrak{h}$, the function $G_k(\tau, s)$ admits an analytic continuation to all $s \in \mathbb{C}$. [You may assume any relevant properties of the function $\Gamma(s)$.]

END OF PAPER