UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 9:00 am to 12:00 pm

PAPER 136

LOCAL FIELDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let L/K be a finite extension of algebraic number fields with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$ respectively.

(i) Define the *inverse different* $\mathcal{D}_{L/K}^{-1}$ of L/K and show that it is a fractional ideal of K whose inverse $\mathcal{D}_{L/K}$ is an ideal in $\mathcal{O}_L$.

(ii) Assume $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ where $\alpha \in \mathcal{O}_L$ and let $g(X) \in K[X]$ be the minimal polynomial for α. Show that $\mathcal{D}_{L/K} = (g'(\alpha))$. Hence determine $\mathcal{D}_{L/\mathbb{Q}}$ for $L = \mathbb{Q}(\sqrt{3})$ and $L = \mathbb{Q}(\sqrt{5})$.

[You may assume without proof that the trace form is non-degenerate for a separable extension of fields.]

(b) Decide whether each of the following rings is a Dedekind domain. Justify your answer.

(i) $\mathbb{Z}[T]$.

(ii) The subring $R := \mathbb{C}[T^2, T^3] \subset \mathbb{C}[T]$ generated by T^2 and T^3.

(iii) $\mathbb{Z}[\zeta_3]$, where ζ_3 is a primitive 3^{rd} root of unity.

2

(a) Let K be a complete non-archimedean valued field. Show that K is locally compact if and only if K is discretely valued and has finite residue field. Deduce that if K is algebraically closed, it cannot be locally compact.

(b)

(i) Show that an absolute value $|\cdot|$ on a field is non-archimedean if and only if $|n|$ is bounded on $\mathbb{Z}$. Deduce that every absolute value on $\mathbb{F}_p(t)$ is non-archimedean.

(ii) Define an absolute value on $\mathbb{F}_3(t)$ such that its completion is isomorphic to $\mathbb{F}_{27}((T))$.

3 (a) Let K be a finite extension of $\mathbb{Q}_p$. Show that there is a finite index subgroup of $\mathcal{O}_K^\times$ which is isomorphic to $(\mathcal{O}_K, +)$.

(b) Let p be an odd prime and let $K = \mathbb{Q}_p(\zeta_p, \sqrt[p]{p})$. Show that $K/\mathbb{Q}_p$ is a totally ramified Galois extension of degree $p(p-1)$. Find a uniformizer for K, and compute the higher ramification groups $G_i(K/\mathbb{Q}_p)$, for $i \in \mathbb{Z}_{\geqslant 0}$.

4

(a) State and prove a version of Hensel's Lemma.

(b) Let K be a local field. Let P be the set of elements $x \in K^\times$ such that x is an m^{th} power for infinitely many integers $m \geqslant 1$. Show that $P = \mathcal{O}_K^\times$.

(c) Determine the number and degrees of the irreducible factors of the polynomial $f(X) = X^4 + 9X^2 - 2$ over $\mathbb{Q}_2$.

5 Let K be a local field with residue $\mathbb{F}_q$ and $\pi \in \mathcal{O}_K$ a uniformizer.

(a) Let $f(X), g(X) \in \mathcal{O}_K[X]$ be Lubin–Tate series for π and let $L(X_1, \ldots, X_n) = \sum_{i=1}^n a_i X_i \in \mathcal{O}_K[X_1, \ldots, X_n]$ be a linear form.

(i) Show that there exists a unique $F(X_1, \ldots, X_n) \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ such that

- $F(X_1, \ldots, X_n) \equiv L(X_1, \ldots, X_n) \mod \deg 2$
- $f \circ F = F \circ g$.

(ii) Let $F(X_1, \ldots, X_n) \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ denote the power series obtained in (i) with $f = g$ and $L(X_1, \ldots, X_n) = \sum_{i=1}^n X_i$. Show that $F(X_1, \ldots, X_n) = F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for any $\sigma \in S_n$, and that for any $a \in \mathcal{O}_K$, there exists $\theta_a(X) \in \mathcal{O}_K[[X]]$ such that $\theta_a(X) \equiv aX \mod X^2$ and $\theta_a \circ F = F \circ \theta_a$.

(b) Let $f(X) = \pi X + X^q$ and let $f_n(X) \in \mathcal{O}_K[X]$ denote the n-fold composition of f. Let α be a root of $f_n(X)$. Show that $K(\alpha)$ is a totally ramified separable extension of K.

END OF PAPER