UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 1:30 pm to 3:30 pm

PAPER 129

INTRODUCTION TO ADDITIVE COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a (not necessarily Abelian) group and let A, B and C be finite subsets of G.

(i) Prove that $|A||BC^{-1}| \leqslant |AB^{-1}||AC^{-1}|$, where XY^{-1} denotes the set $\{xy^{-1} : x \in X, y \in Y\}$.

(ii) Suppose that $|AAA| \leqslant K|A|$. Prove that $|AAAA| \leqslant K^4|A|$. [Hint: you will need to apply the result of the previous part more than once.]

(iii) By considering sets of the form $H \cup \{x\}$, where $|HH|$ is comparable to $|H|$ and x is an element of G, prove that in general if $|AA| \leqslant K|A|$ one cannot deduce that $|AAA|/|A|$ is bounded above by a constant that depends only on K.

2 Let C be a constant, let p be a prime, and let $\phi : \mathbb{F}_p^n \to \mathbb{F}_p^n$ be a function with the property that for every $d \in \mathbb{F}_p^n$ there are at most C distinct values of $\phi(x+d) - \phi(x)$ as x ranges over $\mathbb{F}_p^n$.

(i) Let X be the set of all possible values of $\phi(x) - \phi(x+a) - \phi(x+b) + \phi(x+a+b)$ and let V be a subspace of $\mathbb{F}_p^n$. Prove that if $V \cap X = \{0\}$, then for every translate $W = V + w$ of V the restriction of ϕ to $\phi^{-1}(W)$ is a Freiman homomorphism (of order 2).

(ii) Let Γ be the graph of ϕ. By considering $|(\{0\} \times X) + \Gamma|$ in two different ways, prove that $|X| \leqslant C^5$.

(iii) By considering a random subspace V of suitable codimension and then a random translate of that subspace, prove that there is a subset A of $\mathbb{F}_p^n$ of density at least $p^{-1}C^{-5}$ such that the restriction of ϕ to A is a Freiman homomorphism (again of order 2).

3 (i) State and prove Bogolyubov's lemma.

(ii) Assuming any named theorems from the course that you wish, as well as basic facts about Freiman isomorphisms, prove that for every $c > 0$ and every prime p there exists a constant $c' > 0$ such that if $A \subset \mathbb{F}_p^N$ is a set of size n that contains at least cn^3 additive quadruples, then $2A - 2A$ contains a subspace of size at least $c'n$.

(iii) Assuming any results you like from the course, prove that if A is a subset of $\mathbb{Z}$ of size n and $|A - A| \leqslant Cn$, then A contains an arithmetic progression of length 3 (provided that n is large enough in terms of C).

4 (i) Let G be a finite Abelian group with order not divisible by 2 or 3 and let $f : G \to \mathbb{C}$. Define the U^2 and U^3 norms of f.

(ii) Prove the inequality

$$|\mathbb{E}_{x,d} f_1(x) f_2(x+d) f_3(x+2d) f_4(x+3d)| \leqslant \|f_1\|_2 \|f_2\|_2 \|f_3\|_{U^3} \|f_4\|_{U^3},$$

assuming a similar result for three functions and the U^2 norm. Explain briefly what this result implies about the number of arithmetic progressions of length 4 in a subset A of G.

(iii) Suppose that $\|f\|_{U^3}^8 \geqslant c$ and $\|f\|_\infty \leqslant 1$. Prove that there is a subset $B \subset G$ and a function $\phi : B \to \hat{G}$ such that $|\widehat{\partial_a f}(\phi(a))|^2 \geqslant c/2$ for every $a \in B$ and B^4 contains at least $(c/2)^8 |G|^3$ quadruples (a, b, c, d) such that $a + b = c + d$ and $\phi(a)\phi(b) = \phi(c)\phi(d)$. [You may assume the box-norm inequality.]

(iv) Give a very brief indication how the results of (ii) and (iii) lead to a proof that for every $\delta > 0$ there exists n such that every subset of $\mathbb{F}_5^n$ of density at least δ contains an arithmetic progression of length 4. [For this part there is no need to give proofs – just the basic structure of the argument is enough.]

END OF PAPER