UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 9:00 am to 12:00 pm

PAPER 120

LOGIC AND COMPUTABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Question 1 carries 40 marks. Questions 2 and 3 carry 30 marks each.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Briefly describe the *Curry–Howard correspondence* between intuitionistic propositional logic (IPC) and the simply typed λ-calculus. Your answer should explain what types, type constructors, variables, and simply typed λ-terms correspond to within IPC.

(b) What is a *Heyting algebra*? Show that every Heyting algebra is a distributive lattice.

(c) Provide a Kripke model that does not force the proposition $(\neg\neg p \to q) \vee (\neg\neg q \to p)$. [*You do not need to show that it is a Kripke model.*]

(d) An *implicational proposition* is one containing no logical connectives other than $\to$. The *implicational fragment of intuitionistic propositional calculus* IPC($\to$) admits only implicational propositions, along with the ($\to$)-introduction and ($\to$)-elimination rules and the axiom scheme: $\Gamma, \phi \vdash_{\mathrm{IPC}(\to)} \phi$ for every set of implicational propositions Γ and implicational proposition ϕ.

Prove that IPC($\to$) is complete with respect to Kripke models, i.e., $\Gamma \vdash_{\mathrm{IPC}(\to)} \phi$ iff for all Kripke models $(S, \leqslant, \Vdash)$, the condition $S \Vdash \Gamma$ implies $S \Vdash \phi$.

Let ϕ be an implicational formula and Γ be a set of implicational formulae. Conclude that if $\Gamma \vdash_{\mathrm{IPC}} \phi$, then $\Gamma \vdash_{\mathrm{IPC}(\to)} \phi$.

[*You may use the soundness theorem for the Kripke semantics without proof.*]

2 (a) Let $\mathcal{T}$ be an $\mathcal{L}$-theory and n be a natural number.

Define what an *n-type* of $\mathcal{T}$ is.

Define when such an n-type is *isolated* and when it is *omitted* in an $\mathcal{L}$-structure $\mathcal{M}$.

State the *omitting types theorem.*

(b) Let I be an infinite set. Show that if $\mathcal{U}$ is a non-principal ultrafilter on I and $X \subseteq I$ is finite, then $(I \setminus X) \in \mathcal{U}$.

State *Łoś's theorem.* Use it to construct an example of an infinite field of characteristic p for any given prime number p. [*You may assume the existence of non-principal ultrafilters on any infinite set, as well as any algebraic fact you clearly state, without proof.*]

(c) State the *Ehrenfeucht–Mostowski theorem.*

Show that if $\mathcal{T}$ is a first-order theory admitting infinite models, then $\mathcal{T}$ has models with arbitrarily large group of automorphisms. [*You may use any result from the notes without proof.*]

3 (a) Define the *Church numeral* c_n corresponding to a given natural number n.

What does it mean to say that a function $\mathbb{N}^k \to \mathbb{N}$ is λ-*definable*?

Show that the successor function $\mathrm{succ}\colon \mathbb{N} \to \mathbb{N}$ is λ-definable.

(b) Define what it means for a combinator (i.e., a λ-term without free variables) to be a *fixed point combinator*.

State and prove the *fixed point theorem* for the untyped λ-calculus.

Explain why the fixed point theorem for the untyped λ-calculus does not imply that the successor function $\mathrm{succ}\colon \mathbb{N} \to \mathbb{N}$ has fixed points, even though it is λ-definable.

(c) Show that the set of all fixed point combinators is recursively enumerable.

[*You may assume that the set* Λ^0 *of all combinators is recursively enumerable and use any results from the course without proof.*]

END OF PAPER