UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 1:30 pm to 4:30 pm

PAPER 119

CATEGORY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the terms *monomorphism*, *strong monomorphism* and *regular monomorphism*. Show that any regular monomorphism is strong, and that if two subobjects $A' \rightarrowtail A$, $A'' \rightarrowtail A$ are strong then so is their intersection (= pullback) $A' \cap A'' \rightarrowtail A$, if it exists.

We call a morphism *anodyne* if it is both monic and epic, and we call an object B *saturated* if it is injective with respect to the class of anodyne morphisms, i.e. if every diagram

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d] & A \\ B & \end{tikzcd}$$

where f is anodyne can be completed to a commutative triangle. Show that a strong subobject of a saturated object is saturated.

Now suppose that $\mathcal{C}$ is complete and well-powered, and that every object A of $\mathcal{C}$ admits a monomorphism $A \rightarrowtail B$ with B saturated. Show that every object admits an anodyne morphism to a saturated object [*hint*: consider the intersection of all strong subobjects of B which contain A]. Deduce that the full subcategory $\mathcal{S}$ of saturated objects is reflective in $\mathcal{C}$. Show also that $\mathcal{S}$ is balanced [*hint*: first show that epimorphisms in $\mathcal{S}$ are also epic in $\mathcal{C}$].

2 Recall that a category $\mathcal{C}$ with finite products is said to be *cartesian closed* if the functor $(-) \times A \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A$, for all $A \in \text{ob}\,\mathcal{C}$. A full subcategory $\mathcal{D}$ of a cartesian closed category $\mathcal{C}$ is called an *exponential ideal* if $B \in \text{ob}\,\mathcal{D}$ implies $B^A \in \text{ob}\,\mathcal{D}$ for all $A \in \text{ob}\,\mathcal{C}$. Show that the preorder $\text{Sub}(1)$ of subterminal objects (i.e. objects A such that $A \to 1$ is monic) is an exponential ideal in any cartesian closed category.

Let $\mathcal{D}$ be a reflective subcategory of a cartesian closed category $\mathcal{C}$. Show that $\mathcal{D}$ is an exponential ideal iff the reflector $L \colon \mathcal{C} \to \mathcal{D}$ (that is, the left adjoint of the inclusion) preserves binary products. [*Hint*: an object B belongs to $\mathcal{D}$ iff every morphism $A \to B$ factors uniquely through the unit $A \to LA$.]

Let $\mathbf{2}$ denote the category $(\bullet \to \bullet)$. Show that $[\mathbf{2}, \mathbf{Set}]$ is cartesian closed [*hint*: the exponential $(B_0 \to B_1)^{(A_0 \to A_1)}$ has the form $(F \to B_1^{A_1})$, where F is the set of morphisms $(A_0 \to A_1) \to (B_0 \to B_1)$ and $B_1^{A_1}$ is the exponential in $\mathbf{Set}$]. Deduce that the full subcategory $\text{Mono}(\mathbf{Set})$ of $[\mathbf{2}, \mathbf{Set}]$ whose objects are monomorphisms in $\mathbf{Set}$ is also cartesian closed.

3 A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *final* if, for every $B \in \text{ob}\,\mathcal{D}$, the category $(B \downarrow F)$ is (nonempty and) connected. F is said to be a *discrete fibration* if, for any $A \in \text{ob}\,\mathcal{C}$ and $g\colon B \to FA$ in $\mathcal{D}$, there exists a unique $f \in \text{mor}\,\mathcal{C}$ with $\text{cod}\, f = A$ and $Ff = g$.

(i) Given a commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[d, "F"] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{B} \arrow[r, "K"] & \mathcal{D} \end{tikzcd}$$

where F is final and G is a discrete fibration, show that there is a unique $L\colon \mathcal{B} \to \mathcal{C}$ with $GL = K$ and $LF = H$.

(ii) Show that any functor $F : \mathcal{C} \to \mathcal{D}$ can be factored as a final functor followed by a discrete fibration. [*Hint*: construct a category $\mathcal{E}$ whose objects are the connected components of the categories $(B \downarrow F)$, $B \in \text{ob}\,\mathcal{D}$, and whose morphisms are suitable morphisms of $\mathcal{D}$.]

4 Explain what is meant by a *monad*, and show how every adjunction gives rise to a monad.

Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a full subcategory of the functor category $[\mathcal{C}, \mathcal{C}]$ which is closed under composition and contains the identity functor. If $\mathcal{D}$ has a terminal object T, show that T has a unique monad structure.

Now take $\mathcal{C} = \mathbf{Set}$, and let $\mathcal{D}$ be the category of functors $\mathbf{Set} \to \mathbf{Set}$ which preserve finite coproducts. Show that $\mathcal{D}$ has a terminal object, namely the functor which sends a set A to the set of all ultrafilters on A. [Recall that an ultrafilter on A is a set $U \subseteq PA$ such that (i) $B \in U$ and $B \subseteq B'$ imply $B' \in U$; (ii) $B \in U$ and $B' \in U$ imply $B \cap B' \in U$; and (iii) for every $B \subseteq A$, just one of B and $A \setminus B$ is in U. *Hint*: Given an arbitrary $F \in \text{ob}\,\mathcal{D}$ and $x \in FA$, consider the set of those $B \subseteq A$ for which x is in the image of $FB \to FA$.]

5 Explain carefully what is meant by the statement that filtered colimits commute with finite limits in $\mathbf{Set}$. Give an example to show that this does not hold in $\mathbf{Set}^{\mathrm{op}}$.

Let X be a topological space, and let $\mathcal{O}(X)$ denote the set of open subsets of X, ordered by inclusion. By a *presheaf* (of sets) on X we mean a functor $F\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$. Given a point $x \in X$, we define the *stalk* F_x of F at x to be the colimit of its restriction to $(N_x)^{\mathrm{op}}$, where N_x is the poset of open neighbourhoods of x. Show that the functor $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ sending F to F_x preserves finite limits and has a right adjoint.

A presheaf F is called a *sheaf* if, whenever we are given a family $(U_i \mid i \in I)$ of open sets and a family of elements $s_i \in F(U_i)$ which are compatible in the sense that s_i and s_j have the same image in $F(U_i \cap U_j)$ for each pair (i, j), then there is a unique $s \in F(\bigcup_{i \in I} U_i)$ whose image in $F(U_i)$ is s_i, for each i. We write $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ for the full subcategory of sheaves. Show that the functor $\mathbf{Sh}(X) \to \mathbf{Set}^X$ sending a sheaf F to the family $(F_x \mid x \in X)$ of all its stalks is faithful, and deduce that it is comonadic. [You may assume that $\mathbf{Sh}(X)$ is closed under arbitrary limits in $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, and that it is balanced.]

6 Explain what is meant by a semi-additive category, and prove that binary products and coproducts coincide (in a sense to be explained) if they exist in such a category.

Let $(f, g)\colon A \rightrightarrows B$ be a reflexive parallel pair in an additive category $\mathcal{C}$. Show that (f, g) has the structure of an internal groupoid in $\mathcal{C}$: that is, for any object C, the elements of $\mathcal{C}\,(C, B)$ are the objects of a groupoid whose morphisms are the elements of $\mathcal{C}\,(C, A)$, with domain and codomain operations given by composition with f and g respectively. [*Hint*: the composite of a pair $(x, y)\colon C \rightrightarrows B$ with $gx = fy$ is $x + y - rgx$, where r is a common splitting for f and g.]

By considering the usual order relation on $\mathbb{N}$ as an object in the category of commutative monoids, or otherwise, show that we cannot weaken 'additive' to 'semi-additive' in the previous paragraph.

END OF PAPER