UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 9:00 am to 12:00 pm

PAPER 118

COMPLEX MANIFOLDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define an *almost complex structure* J on a smooth manifold M. Given an almost complex structure, define what is meant by the *differential* (p,q)*-forms* on M and by the the differential operators ∂ and $\bar{\partial}$ on complex differential forms on M.

Prove that the following three statements are equivalent:

(i) if X, Y are vector fields of type $(1,0)$ on M, then the vector field $[X,Y]$ is also of type $(1,0)$;

(ii) $d\alpha = \partial\alpha + \bar{\partial}\alpha$ for each complex 1-form α on M;

(iii) $\bar{\partial}\bar{\partial}f = 0$ for every complex-valued function f on M.

An almost complex structure J is said to be *integrable* if any of (i), (ii), (iii) holds.

If M is a complex manifold, explain what is meant by the almost complex structure J_M induced by the holomorphic atlas, showing that J_M is independent of the choice of local complex coordinates and integrable.

Let φ be a $(0,q)$-form on $\mathbb{C}^n$, $q > 0$, such that $\bar{\partial}\varphi = 0$. Show that for each bounded polydisc there is a $(0,q-1)$-form ψ on this polydisc such that $\varphi = \bar{\partial}\psi$.

Explain briefly why the form $d\bar{z}$ on $\mathbb{C}$ induces a well-defined $(0,1)$-form on the quotient manifold $\mathbb{C}/(z \sim z+i)$, $z \in \mathbb{C}$. Is this latter $(0,1)$-form $\bar{\partial}$-exact on $\mathbb{C}/(z \sim z+i)$? Justify your answer.

[*Basic relations between real differential forms and real vector fields may be assumed if accurately stated.*
You may assume that if g is a smooth function on $\mathbb{C}$ and $D \subset \mathbb{C}$ is an open disc, then $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_D \frac{g(w)}{w-z}\,dw \wedge d\bar{w}$ *is smooth and* $\dfrac{\partial f}{\partial \bar{z}} = g$ holds on D.]

2 Define the terms *holomorphic line bundle* L over a complex manifold X and *holomorphic local section* of L. Suppose that a holomorphic line bundle L is endowed with a Hermitian inner product h on the fibres. Define what is meant by the *Chern connection* on L. Prove the existence and uniqueness of a Chern connection.

Prove a formula for the curvature form $F(A)$ of the Chern connection in terms of non-vanishing local holomorphic sections of L, showing that $F(A)$ is a $(1,1)$-form which is well-defined globally over S.

Show further that if $\hat{h}$ is another Hermitian inner product on the fibres of L and $\hat{A}$ is the corresponding Chern connection, then $F(\hat{A}) - F(A) = \bar{\partial}\partial u$, for some smooth function u on X.

Now let L_1, L_2 be two holomorphic line bundles and let A_1, A_2 be connections on respectively L_1, L_2. Explain how A_1 and A_2 induce a connection $A_1 \otimes A_2$ on $L_1 \otimes L_2$ with curvature $F(A_1) + F(A_2)$. Show that if each A_i is the Chern connection for a Hermitian inner product h_i on the fibres of L_i, then $A_1 \otimes A_2$ is the Chern connection for an appropriate Hermitian inner product h on $L_1 \otimes L_2$ which you should determine.

[*Standard properties of connections on complex vector bundles over smooth manifolds may be assumed if accurately stated.*]

3 Define the *tautological bundle* $\mathcal{O}(-1)$ over $\mathbb{C}P^n$ and show that it has holomorphic transition functions (for appropriate local trivializations). Define, for an arbitrary integer k, the holomorphic line bundle $\mathcal{O}(k)$ over $\mathbb{C}P^n$. State the relation between transition functions for $\mathcal{O}(-1)$ and transition functions for $\mathcal{O}(k)$.

Show that every hyperplane H in $\mathbb{C}P^n$ is a divisor of $\mathcal{O}(1)$. By considering appropriate meromorphic functions on $\mathbb{C}P^n$, or otherwise, show that every effective divisor of $\mathcal{O}(1)$ is given by a hyperplane in $\mathbb{C}P^n$.

[*Basic properties of divisors of holomorphic line bundles may be assumed if accurately stated.*
If needed, you may assume that if $f : \mathbb{C}^n \to \mathbb{C}$ *is a holomorphic function such that* $|f(a+bz)| < C \max\{1, |z|\}$ *for all* $z \in \mathbb{C}$ *and* $a, b \in \mathbb{C}^n$*, with a constant* $C > 0$ *independent of* z*, then* f *is a linear function.*]

Define the *blow-up* $\sigma : \widetilde{X} \to X$ of a complex manifold X, $\dim_{\mathbb{C}} X > 1$, at a point $x \in X$. Show that $\widetilde{X}$ is a complex manifold and σ is a holomorphic map which are (up to biholomorphic equivalence) independent of a choice of coordinates.

Now let $X = \mathbb{C}^2$ and $x = 0$ and consider the map $a(z) = -z$, $z \in \mathbb{C}^2$. Prove that a lifts to a holomorphic map $\tilde{a} : \widetilde{\mathbb{C}}^2 \to \widetilde{\mathbb{C}}^2$ such that $a \circ \sigma = \sigma \circ \tilde{a}$ and the quotient $S = \widetilde{\mathbb{C}}^2/(p \sim \tilde{a}(p))$, $p \in \widetilde{\mathbb{C}}^2$ is a complex manifold. Show further that S is biholomorphic to the total space of the line bundle $\mathcal{O}(-2)$ over $\mathbb{C}P^1$.

4 Let (X, h) be a Hermitian manifold. Define the *fundamental form* ω of the Hermitian metric h and show that ω is a real $(1,1)$-form. Show that the volume form of h can be expressed as $\omega^n/n!$, where $n = \dim_{\mathbb{C}}(X)$. Define the *Hodge $*$-operator* and show that $*\omega = \omega^{n-1}/(n-1)!$.

Define the *Laplacian* $\Delta = \Delta_d$ and the *complex Laplacian* $\Delta_{\bar\partial}$ on X. State the Hodge theorem for the space of (p,q)-forms. Show that if a (p,q)-form η on a compact Hermitian manifold satisfies $\Delta_{\bar\partial}\eta = 0$, then η is $\bar\partial$-closed and is not $\bar\partial$-exact unless $\eta = 0$.

Now let X be a compact Kähler manifold. Prove that $d^c d^* + d^* d^c = 0$. Suppose that a complex k-form $\alpha \in \Omega^k(X)$ is d-closed and d^c-exact. Prove that there exists a form $\beta \in \Omega^{k-2}(X)$ such that $\alpha = dd^c\beta$.

[*You may assume that the formal* L^2*-adjoint of* $\bar\partial$ *on a Hermitian manifold can be expressed as* $\bar\partial^* = - * \partial *$*. The identities* $[\bar\partial^*, L] = i\partial$ *and* $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$ *on Kähler manifolds may be assumed, if you define* L*.*]

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