UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 1:30 pm to 3:30 pm

PAPER 116

LARGE CARDINALS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
Question 1 carries 40 marks; question 2 carries 60 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Each of the two parts of this question gives a mathematical statement with two underlined terms.

For each of them, first give a precise and correct definition of the two underlined terms (five marks for each defined term); then explain why the statement holds (ten marks for each statement).

In your explanation, you are not required to give proof details; instead, explain the proof structure of the result stated by providing mathematically precise and correct statements of theorems proved in the lectures and arguing how the result stated follows from them.

(a) If κ is a *weakly compact cardinal*, then there is an *inaccessible cardinal* smaller than κ.

(b) Every *measurable cardinal* is a 1-*strong cardinal*.

2

(a) Prove that every measurable cardinal is a strong limit cardinal.

In parts (b) through (f), you may use theorems proved in the lectures provided that you state them clearly and precisely.

(b) We say that a first order formula φ *describes* an ordinal α if α is the least ordinal such that $\mathbf{V}_\alpha \models \varphi$. Show that if κ is inaccessible there is no first order formula that describes κ.

(c) If Φ is a cardinal property, we write $\Phi\mathsf{C}$ for the statement "there is a cardinal κ such that $\Phi(\kappa)$"; if $\Phi\mathsf{C}$ holds, we write ι_Φ for the least cardinal κ such that $\Phi(\kappa)$.

As in the lectures, we write $\Phi <_1 \Psi$ if and only if $\mathsf{ZFC} + \Phi\mathsf{C} + \Psi\mathsf{C}$ implies that $\iota_\Phi < \iota_\Psi$. Assuming appropriate consistency statements, show that the relation $<_1$ is not transitive, i.e., there are cardinal properties Φ_0, Φ_1, and Φ_2 such that $\Phi_0 <_1 \Phi_1$, $\Phi_1 <_1 \Phi_2$, but $\Phi_0 \not<_1 \Phi_2$.

(d) Let

$$T := \mathsf{ZFC} + \text{“there is a worldly cardinal” and}$$
$$T^* := \mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC}).$$

Assume that both theories are consistent and show that T has strictly greater consistency strength than T^*.

For parts (e) and (f), assume that $\kappa_0 < \kappa_1 < \lambda$ are such that κ_0 and κ_1 are measurable and λ is inaccessible. Let U_0 be a κ_0-complete ultrafilter on κ_0, U_1 be a κ_1-complete ultrafilter on κ_1, and $j_0 : \mathbf{V}_\lambda \to M_0$ and $j_1 : \mathbf{V}_\lambda \to M_1$ be the ultrapower embeddings into transitive models M_0 and M_1 built from U_0 and U_1, respectively.

(e) Determine $j_0(\kappa_1)$ and $j_1(\kappa_0)$.

(f) Assume that $\mathbf{V}_\lambda \models \mathsf{GCH}$. Show that neither $j_0(\kappa_0)$ nor $j_1(\kappa_1)$ is a cardinal in $\mathbf{V}_\lambda$.

END OF PAPER