UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2022 1:30 pm to 4:30 pm

PAPER 115

DIFFERENTIAL GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Fix positive integers m and n with $m \leqslant n$, and let $X_{m,n}$ be the set of $m \times n$ real matrices of rank m. Given a multi-index $I = (i_1, \ldots, i_m)$ from $\{1, \ldots, n\}$, and an $m \times n$ matrix M, let M_I be the $m \times m$ matrix made up of the columns of M labelled by I.

(a) By considering the sets U_I, defined by

$$U_I = \{M \in X_{m,n} : M_I \text{ is invertible}\},$$

show that $X_{m,n}$ is naturally a smooth manifold of dimension mn.

Let $E_{m,n} = \{(M, \mathsf{v}) \in X_{m,n} \times \mathbb{R}^n : \mathsf{v} \in \ker M\}$, and let $\pi : E_{m,n} \to X_{m,n}$ be the map $(M, \mathsf{v}) \mapsto M$.

(b) Show that $E_{m,n}$ is a submanifold of $X_{m,n} \times \mathbb{R}^n$.

(c) Prove that $\pi : E_{m,n} \to X_{m,n}$ is a vector bundle of rank $n - m$. [*Hint: You may find it helpful to decompose M into M_I and the remaining columns $M_{I'}$, and similarly to decompose v into the entries v_I labelled by I and the remaining entries $\mathsf{v}_{I'}$.*]

(d) Write down a natural smooth map $F : S^2 \to X_{1,3}$ and explain briefly why $F^*E_{1,3}$ is isomorphic to TS^2. Assuming that TS^2 does not admit a nowhere-zero section, deduce that $E_{1,3}$ is non-trivial.

2 Let X be an oriented n-manifold-with-boundary, and let $F : \partial X \to X$ denote the inclusion map.

(a) State and prove Stokes's theorem for X, explaining clearly how ∂X is oriented. You should comment briefly on why any sums you write down make sense.

Now suppose X carries a Riemannian metric g, and equip ∂X with the metric F^*g. Let ω_X and $\omega_{\partial X}$ denote the positively oriented unit volume forms on X and ∂X.

Given a point $p \in \partial X$, let $\mathsf{v}_1, \ldots, \mathsf{v}_n$ be positively oriented orthonormal vector fields on X defined in a neighbourhood U of p. Assume that on ∂U the vector field v_1 is orthogonal to ∂X and points outwards.

(b) By considering the dual 1-forms, show that on ∂U we have

$$\omega_{\partial X} = F^*(\iota_{\mathsf{v}_1} \omega_X).$$

(c) Let X be the closed unit ball D^n in $\mathbb{R}^n$ with the Euclidean metric $\sum_{i=1}^n (\mathrm{d}x^i)^2$. Write down coordinate expressions for ω_X and $\omega_{\partial X}$ in terms of the x^i. Deduce that

$$\int_{\partial X} \omega_{\partial X} = n \int_X \omega_X,$$

i.e. that the $(n-1)$-dimensional volume of the unit $(n-1)$-sphere is n times the n-dimensional volume of the ball it bounds.

3 Let v be a vector field on a manifold X.

(a) Define a *local flow* of v, and the *Lie derivative* of a differential form α along v. State Cartan's magic formula.

Now let G be a Lie group. For $g \in G$ let L_g and R_g denote left- and right-translation by g respectively, and for $\xi \in \mathfrak{g}$ let l_ξ be the associated left-invariant vector field.

(b) Show that $\Phi^t = R_{\exp(t\xi)}$ defines a global flow of l_ξ, and hence that a differential form α on G satisfies $\mathcal{L}_{\mathsf{l}_\xi}\alpha = 0$ for all ξ if and only if the map

$$f : G \to \Omega^\bullet(G) \quad \text{given by} \quad f(g) = R_g^*\alpha$$

is locally constant. [You may assume basic properties of the exponential map.]

(c) Suppose that G is connected, and let α be a left-invariant 1-form on G. Show that α is closed if and only if it's bi-invariant. Must this still be true if we drop the connectedness condition?

4 (a) State the differential forms version of the Frobenius integrability theorem and give, with brief justification, examples of integrable and non-integrable 2-plane distributions on $\mathbb{R}^3$.

Let $\mathcal{A}$ be a connection on a principal G-bundle $\pi : P \to B$.

(b) Define the *horizontal distribution* H and curvature $\mathcal{F}$ of $\mathcal{A}$, and show that H is integrable if and only if $\mathcal{F} = 0$.

Now let B be the cylinder $S^1 \times \mathbb{R}$ with local coordinates (θ, t), and let P be the trivial $\mathbb{R}$-bundle with fibre coordinate z. Consider the distribution D on P spanned by $\partial_\theta + f\partial_z$ and $\partial_t + g\partial_z$, where f and g are smooth functions of θ, t, and z.

(c) State and prove necessary and sufficient conditions on f and g for D to be the horizontal distribution of a connection $\mathcal{A}$ on P. If these conditions hold, compute $\mathcal{F}$.

(d) For $f = t\cos\theta + 1$ and $g = \sin\theta$, does the connection admit local horizontal sections? What about global horizontal sections?

END OF PAPER