UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 9:00 am to 12:00 pm

PAPER 114

ALGEBRAIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain, without making use of cellular homology, the calculation of $H_i(S^n;\mathbb{Z})$ for all $n > 0$ and all $i \geqslant 0$, and hence determine the effect on homology of a reflection $r : S^n \to S^n$. *[You may use any other standard properties of homology, provided they are clearly stated.]*

Define the cellular chain complex of a CW-complex X, and describe it in terms of the cells of X and the attaching maps of these cells.

Describe a CW-complex structure on $\mathbb{RP}^n$ and hence calculate the cellular homology of this space, with both $\mathbb{Z}$- and $\mathbb{Z}/2$-coefficients.

2 Let $(C_\bullet, d_C)$ be a chain complex. For a $\mathbb{Z}$-module A write $H_*(C_\bullet; A)$ for the homology of the chain complex $C_\bullet \otimes A$ and $H^*(C_\bullet; A)$ for the homology of the chain complex $\mathrm{Hom}_{\mathbb{Z}}(C_\bullet, A)$, in both cases with the differential induced by d_C.

Assume that each C_i is a finitely-generated free $\mathbb{Z}$-module. By explaining how to decompose $(C_\bullet, d_C)$ as a sum of elementary chain complexes, describe $H^*(C_\bullet;\mathbb{Z})$ in terms of $H_*(C_\bullet;\mathbb{Z})$.

Show that if $H_*(C_\bullet;\mathbb{Z}/p) = 0$ for all prime numbers p, then $H_*(C_\bullet;\mathbb{Z}) = 0$ too.

Let $f_\# : (C_\bullet, d_C) \to (D_\bullet, d_D)$ be a map between chain complexes of finitely-generated free $\mathbb{Z}$-modules. Verify that

$$M_i := C_{i-1} \oplus D_i, \qquad d_M := \begin{pmatrix} -d_C & 0 \\ -f_\# & d_D \end{pmatrix}$$

defines a chain complex, and hence show that if $f_* : H_*(C_\bullet;\mathbb{Z}/p) \to H_*(D_\bullet;\mathbb{Z}/p)$ is an isomorphism for all prime numbers p, then $f_* : H_*(C_\bullet;\mathbb{Z}) \to H_*(D_\bullet;\mathbb{Z})$ is an isomorphism too.

3 Let R be a commutative ring. What is an R-orientation of a vector bundle? Prove that a complex vector bundle has a canonical R-orientation.

Carefully state the Thom isomorphism theorem for an R-oriented vector bundle $\pi : E \to B$ over a compact space B. Define the Euler class of such a vector bundle, and derive its Gysin exact sequence. Calculate the $\mathbb{Z}$-cohomology ring of $\mathbb{CP}^n$.

Carefully state the Künneth theorem. Writing $X := \mathbb{CP}^n \times \mathbb{CP}^n$, determine the group of automorphisms of $H^2(X;\mathbb{Z})$ which may be realised by homotopy equivalences $f : X \to X$.

4 Let R be a commutative ring, and M be a compact d-dimensional manifold. What is a local R-orientation of M at a point $x \in M$? What is an R-orientation of M? What is an R-fundamental class (i.e. with R-coefficients) of M? Show that an R-fundamental class of M determines an R-orientation. *[You do not need to show that an orientation determines a fundamental class.]*

State the Poincaré duality theorem for M.

Assuming the calculation of the $\mathbb{Z}$-homology of spheres, show that S^d for $d \geqslant 1$ has a $\mathbb{Z}$-fundamental class.

If $X \subset S^d$ is a proper subspace such that the quotient space S^d/X is a manifold, show that

1. S^d/X has a $\mathbb{Z}$-orientation such that the quotient map $q : S^d \to S^d/X$ has degree 1,
2. $S^d - X$ has the $\mathbb{Z}$-homology of a point.

END OF PAPER