UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 1:30 pm to 4:30 pm

PAPER 113

ALGEBRAIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define *properness* for a morphism of schemes and state the valuative criterion for properness. Explicitly verify the valuative criterion for the structure morphism $\mathbb{P}^2_k \to \mathrm{Spec}\ k$ for maps from discrete valuation rings.

(b) Give an example of a scheme X over Spec $\mathbb{C}$ that is universally closed but fails to be proper.

(c) Let X be a Noetherian scheme that is proper over a field k. Let $\mathcal{A}_\bullet$ be a coherent sheaf of graded $\mathcal{O}_X$-algebras that is locally generated in degree 1. Define Y to be the global Proj construction:

$$Y := \underline{\mathrm{Proj}}_X \mathcal{A}_\bullet.$$

Prove that Y is proper over k.

(d) Let X and Y be schemes over Spec k. Let $f, g : X \to Y$ be morphisms. Prove that there exists a largest locally closed subscheme of X such that f and g coincide on this subscheme. If Y is separated, prove that this locus is a closed subscheme of X.

2

A scheme over a field k will be said to satisfy condition ($\star$) if it is noetherian, integral, separated, and regular in codimension 1.

(a) Let X be a scheme over a field k satisfying ($\star$). Define a Weil divisor on X. Define the class group of X.

(b) Let X be $\mathbb{P}^3_k$ and let C be a closed subscheme of dimension 1. Show that $U = X \setminus C$ satisfies ($\star$). Calculate the class group of the scheme U.

(c) Prove or give a counterexample: every closed subscheme of affine space satisfying ($\star$) has torsion free class group.

(d) Let $\iota : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be the Segre embedding, with image given by the scheme theoretic vanishing of $XZ - YW$ in the homogeneous coordinates on $\mathbb{P}^3$. Prove that both $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^3$ are regular, i.e. the local rings at all points are regular. Prove that pulling back line bundles gives rise to a well-defined map of divisor class groups

$$\iota^\star : Cl(\mathbb{P}^3) \to Cl(\mathbb{P}^1 \times \mathbb{P}^1).$$

Calculate the domain, target, and image of this map. [You may use the fact that local rings of $\mathbb{A}^n$ are regular.]

3

Let k be a field.

(a) Let Z be an integral closed subscheme of $\mathbb{A}^2_k$. Let U be the open subscheme $\mathbb{A}^2_k \setminus Z$. Calculate the sheaf cohomology group $H^0(U, \mathcal{O}_U)$ and comment on how the answer depends on the dimension of Z.

(b) Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups. Fix an open cover of X and define the Čech cohomology groups with respect to this open cover. By choosing an appropriate cover, explicitly calculate the sheaf cohomology group $H^1(\mathbb{P}^1_k, \mathcal{O}(-1))$.

(c) Let X be the scheme

$$\operatorname{Spec} k[X_1, X_2, X_3] \setminus \{\langle X_1, X_2, X_3 \rangle\} = \mathbb{A}^3_k \setminus \{\underline{0}\}.$$

By using Čech cohomology for an appropriate open cover, calculate all sheaf cohomology groups for the sheaf $\mathcal{O}_X$.

(d) Let f_d be a homogeneous polynomial in 4 variables of degree d. Let X be the scheme theoretic vanishing locus of f_d and consider

$$i : X \hookrightarrow \mathbb{P}^3_k.$$

Describe an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to i_\star \mathcal{O}_X \to 0.$$

Deduce that $H^1(X, \mathcal{O}_X)$ vanishes. [You may use any result about the cohomology of sheaves on projective space provided it is clearly stated].

END OF PAPER