UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 1:30 pm to 4:30 pm

PAPER 112

KNOTS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For each item below, give an example of a knot with the stated properties. Justify your answers. You may use theorems proved in lecture, as long as you state them clearly.

(i) A knot K_1 with $g(K_1) = 2022$.

(ii) A knot K_2 with $g(K_2) = 1$ and $c(K_2) = 2022$.

(iii) A knot K_3 for which $\Delta_{K_3}(t) \not\sim 1$, but $\det K_3 = |\Delta_{K_3}(-1)| = 1$.

(iv) A knot K_4 which is not isotopic to the unknot, but which has $\Delta_{K_4}(t) \sim 1$. [You should explain why $\Delta_{K_4}(t) \sim 1$, but do not need to justify that your example is not the unknot.]

2 Consider two links L and L' represented by diagrams D and D' of the form shown in the figures below. Here D_1 and $r(D_1)$, the parts of D and D' contained in the left-hand disks, are related by a $180°$ rotation around a vertical line as shown.

(a) Prove that L and L' can be oriented so that $V(L) = V(L')$. [*Hint: Use the Kauffman bracket skein relation to resolve all crossings in* D_1.]

(b) Show that the links L and L' need not be isotopic to each other. [*Hint: Look for an example where the individual components of* L *and* L' *are different knots.*]

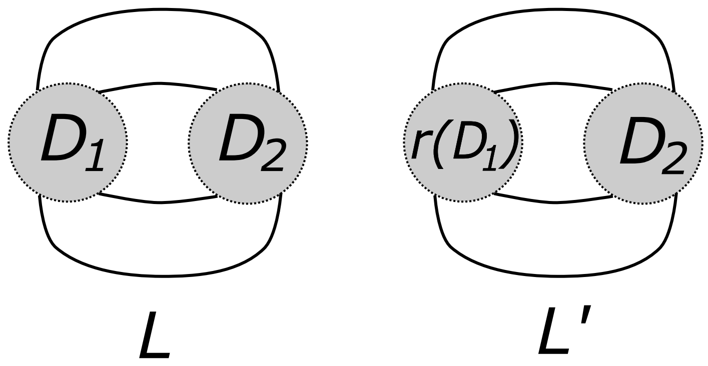

Figure 1: The diagrams D and D'.

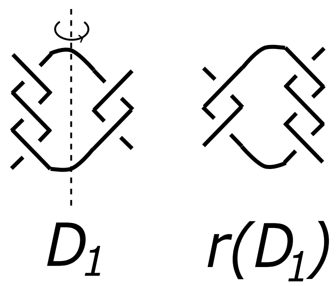

Figure 2: An example of $180°$ rotation.

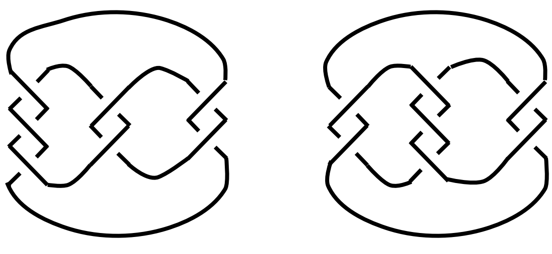

Figure 3: An example of D and D'.

3 Let K be a knot represented by a diagram D, with complementary regions R_0 (the infinite region), $R_1, \ldots, R_n$. Assume that D is reduced, so that no two corners of a crossing of D belong to the same region R_i. Let R_0 be the infinite region, and let R_1 be a region of D which is adjacent to R_0, as shown in the figures below. Let

$$P_{Dehn} = \langle a_1, \ldots, a_n \,|\, w_1, \ldots, w_{n-1} \rangle$$

be the *Dehn presentation* of $\pi_1(E_K)$ associated to D.

(a) Explain how to find the relations w_j from D. [No justification is required.]

(b) Let a_1 be the generator associated to the region R_1. Explain how to compute the Alexander polynomial $\Delta_K(t)$ from the Fox derivatives $d_{a_i} w_j$ for $i > 1$. Justify your answer.

(c) A *Kauffman state* for D is a decoration of D in which we draw a black dot in one of the four corners adjacent to each crossing of D, subject to the constraint that R_0 and R_1 contain no dots, and that for $i > 1$, R_i should contain exactly one dot. See the figure for two examples of Kauffman states associated to a diagram of the figure 8 knot.

Let $\mathcal{S}(D)$ be the set of all Kauffman states of D. Show that

$$\Delta_K(t) \sim \sum_{s \in \mathcal{S}(D)} (-1)^{\epsilon(s)} t^{\delta(s)}$$

where $\epsilon(s)$ and $\delta(s)$ are integers associated to s. [You do not need to describe how to find $\epsilon(s)$ and $\delta(s)$.]

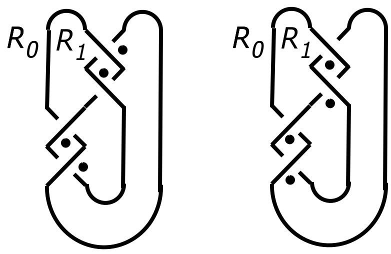

Figure 1: Two Kauffman states for a diagram of the figure-eight knot.

4 Let $K \subset S^3$ be a knot. Briefly explain why its exterior E_K is homotopy equivalent to a cell complex X with 1 0-cell, n 1-cells, and $n-1$ 2-cells for some number $n \geqslant 0$. [A detailed proof is not required.]

Show that there is a unique surjective homomorphism $\alpha : \pi_1(X) \to \mathbb{Z}/2$. Let $p : \widehat{X} \to X$ be the covering map corresponding to $\ker \alpha$. Show that $H_*(\widehat{X})$ has the structure of a module over $\widehat{R} = \mathbb{Z}[\mathbb{Z}/2] \cong \mathbb{Z}[t]/(t^2-1)$. Briefly explain how to give $\widehat{X}$ a cell structure so that $C_*^{cell}(\widehat{X})$ is a free module over $\widehat{R}$. How is $C_*^{cell}(\widehat{X})$ related to $C_*^{cell}(\widetilde{X})$, where $\widetilde{X}$ is the infinite cyclic cover of X?

Suppose that p is an odd prime, and that $\mathbb{F}_p$ is the field of order p. Show that $H_*(\widehat{X};\mathbb{F}_p) \cong H_*(E_K;\mathbb{F}_p) \oplus H_*(C_-)$, where $C_- = C_*^{cell}(\widetilde{X}) \otimes_R R_-$, $R = \mathbb{Z}[t^{\pm 1}]$, and $R_- = \mathbb{F}_p[t^{\pm 1}]/(t+1)$. By considering the Alexander polynomial or otherwise, prove that $H_*(C_-)$ is nonzero if and only if p divides $\det K$. What is $H_*(\widehat{X};\mathbb{Q})$?

5 (a) Suppose $C : S^{k-1} \hookrightarrow N^{n-1}$ is a smoothly embedded sphere in a manifold N of dimension $n-1$. What is meant by a *framing* of C? Give an example of such a C which has no framings. If a framing exists, describe the set of homotopy classes of framings of C. [No justification is needed.] What is this set when $k=2$ and $n=4$?

(b) Let $\widehat{L}_1$ be the (n,n) torus link with framing 1 (relative to the Seifert framing) on each component. What is the intersection form on $W(\widehat{L_1})$? Compute $H_*(S^3_{\widehat{L}_1})$. Identify the manifold $W(\widehat{L_1})$ and its boundary $S^3_{\widehat{L}_1}$.

(c) Now let $\widehat{L}_2$ be the (n,n) torus link with framing 0 (relative to the Seifert framing) on each component. What is the intersection form on $W(\widehat{L_2})$? Compute $H_*(S^3_{\widehat{L}_2})$ and identify the manifold $S^3_{\widehat{L}_2}$.

END OF PAPER