UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 1:30 pm to 3:30 pm

PAPER 111

COXETER GROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let W be a group generated by a finite set S of involutions.

(a) Write down the condition required for the pair (W, S) to satisfy the *deletion condition* (D) and the condition required for the pair (W, S) to satisfy the *exchange condition* (E).

The pair (W, S) satisfies the *folding condition* (F), if the following condition holds: suppose $w \in W$ and $s, t \in S$ are such that $l_S(sw) = l_S(w) + 1$ and $l_S(wt) = l_S(w) + 1$. Then either $l_S(swt) = l_S(w) + 2$ or $swt = w$.

(b) Show that the conditions (D), (E), and (F) are equivalent, by proving (D) $\Rightarrow$ (E) $\Rightarrow$ (F) $\Rightarrow$ (D).

2 Let (W, S) be a Coxeter system, and $R = \{wsw^{-1} \mid w \in W,\ s \in S\}$.

(a) Define the Cayley graph for W with generating set S, $\mathrm{Cay}_S(W)$, and the wall H_r associated to a reflection $r \in R$.

(b) Given a reflection $r \in R$ and element $w \in W$, show that H_r separates e from w in $\mathrm{Cay}_S(W)$ if and only if $l_S(rw) < l_S(w)$.

(c) Show that if an element $w_0 \in W$ satisfies that $l_S(rw_0) < l_S(w_0)$ for all $r \in R$, it follows that for all $u \in W$, $l_S(w_0) = l_S(u) + l_S(u^{-1}w_0)$.

3 Let (W, S) be a Coxeter system, $T \subseteq S$ and $W_T = \langle T \rangle$ the subgroup of W generated by T.

(a) Show that (W_T, T) is a Coxeter system.

(b) Show that for $w \in W_T$, any reduced word $(s_1, \ldots, s_k)$ representing w in W satisfies $s_i \in T$ for all i.

(c) Show the following inclusions of subgroups in Coxeter groups:

$$W(A_n) \subset W(B_{n+1}),\ W(A_7) \subset W(E_8),\ W(D_7) \subset W(E_8).$$

(d) Without using the classification of finite Coxeter groups, show that the Coxeter group

$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^4 = (tu)^4 = (su)^2 = e\rangle$$

is not a subgroup of the finite Coxeter group $W(B_4)$.

4 Let (W,S) be a Coxeter system. Recall that the Davis complex $\Sigma(W,S)$ can be defined in two equivalent ways, as:

(i) A basic construction $\mathcal{U}(W,K)$.

(ii) The geometric realisation $Flag(\mathcal{P})$ of the poset $\mathcal{P} = \{wW_T | w \in W,\, T \in \mathcal{S}\}$.

(a) Draw the Davis complex $\Sigma(W,S)$ twice for the Coxeter system with $S = \{s,t\}$ and $m_{st} = 2$, first using definition (i) and then using definition (ii).

(b) Let (W,S) be a Coxeter system. Show using definition (i) that for every $w \in W$ and every $T \in \mathcal{S}$, the conjugate wW_Tw^{-1} appears as the stabiliser of some vertex of the Davis complex $\Sigma(W,S)$.

(c) Let (W,S) be a Coxeter system. Prove that the Davis complex $\Sigma(W,S)$ is contractible. You can use the following claims without proof:

C1: If there exists $w_0 \in W$ such that $l_S(sw_0) < l_S(w_0)$ for all $s \in S$ then W is finite.

C2: For every parabolic subgroup W_T of W, there exists a unique element $w \in W$ of minimal length in the coset wW_T, such that all elements $w' \in wW_T$ can be written as $w' = wa$ for $a \in W_T$ with $l_S(w') = l_S(w) + l_S(a)$.

C3: If $T \in \mathcal{S}$, then $K^T = \cup_{t \in T} K_t$ is contractible. (Here K_t is the t-mirror in the fundamental chamber K.)

C4: The union of two contractible spaces with contractible intersection is contractible.

END OF PAPER