UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 9:00 am to 11:00 am

PAPER 109

COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Local LYM inequality. State the LYM inequality, and give two proofs: one using Local LYM and one using maximal chains.

For a set $A \in \mathcal{P}([n])$, we write $\mu(A)$ for $(-1)^{|A|}$. A family $\mathcal{A} \subset \mathcal{P}([n])$ is called *convex* if whenever $A, B \in \mathcal{A}$ and $A \subset C \subset B$ then also $C \in \mathcal{A}$. Show that if $\mathcal{A}$ is convex then

$$\left| \sum_{A \in \mathcal{A}} \mu(A) \right| \leqslant \binom{n}{\lfloor n/2 \rfloor}.$$

2

Show that every maximal intersecting family in $\mathcal{P}([n])$ has size exactly 2^{n-1}. For each $n \geqslant 4$, exhibit three intersecting families of size 2^{n-1} that are non-isomorphic.

State the Erdős-Ko-Rado theorem, and give two proofs: one using the Kruskal-Katona theorem and one using averaging.

Let $f_r(n)$ be the *minimal* size of a maximal intersecting family in $[n]^{(r)}$. For a fixed $r \geqslant 2$, does $f_r(n)$ tend to infinity as n tends to infinity?

3

State and prove the vertex-isoperimetric inequality in the grid $[k]^n$.

Determine which of the following statements are true and which are false.

(i) For k sufficiently large, if A and B are disjoint subsets of $[k]^2$ of size greater than $(k^2 - k)/2$ then some point of A is adjacent to some point of B.

(ii) For k sufficiently large, if A and B are disjoint subsets of $[k]^3$ of size greater than $(k^3 - k^2)/2$ then some point of A is adjacent to some point of B.

(iii) For k sufficiently large (and a multiple of 3), if A is a subset of $[k]^2$ of size $4k^2/9$ then there are at least $4k/3$ edges from A to its complement.

4

State and prove the Frankl-Wilson theorem.

In each of the following cases, determine the maximum size of the given family, up to a multiplicative constant.

(i) A family $\mathcal{A} \subset [n]^{(4)}$ such that for any distinct $A, B \in \mathcal{A}$ we have that $|A \cap B|$ is 2 or 3.

(ii) A family $\mathcal{A} \subset [n]^{(4)}$ such that for any distinct $A, B \in \mathcal{A}$ we have that $|A \cap B|$ is 1 or 3.

(iii) A family $\mathcal{A} \subset [n]^{(4)}$ such that for any distinct $A, B \in \mathcal{A}$ we have that $|A \cap B|$ is 0 or 2.

(iv) A family $\mathcal{A} \subset [n]^{(4)}$ such that for any distinct $A, B \in \mathcal{A}$ we have that $|A \cap B|$ is 1 or 2.

END OF PAPER