UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 9:00 am to 12:00 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $c : \Omega \to \mathbb{R}$ is a given function.

(a) Suppose $c \leqslant 0$. State and prove the weak maximum principle satisfied by a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\Delta u + cu \geqslant 0$ in Ω. Use the weak maximum principle to prove the following:

(i) if $c \leqslant 0$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta u + cu = f$ in Ω for some function $f : \Omega \to \mathbb{R}$, then for any constant $d > 0$ such that $\Omega \subset \{-d < x_1 < d\}$, we have

$$\sup_{\Omega} |u| \leqslant \sup_{\partial \Omega} |u| + (e^{2d} - 1) \sup_{\Omega} |f|.$$

[*Hint: consider a function v of the form $v(x_1, x_2, \ldots, x_n) = A + B(e^{2d} - e^{x_1 + d})$ for appropriate constants A and B.*]

(ii) if $\varphi \in C^0(\overline{\Omega})$, $g \in C^1(\Omega \times \mathbb{R}; \mathbb{R})$ and g satisfies $\frac{\partial g}{\partial t}(x,t) \geqslant 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, then there is at most one function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\Delta\, u = g(x,u)$ in Ω and $u = \varphi$ on $\partial\, \Omega$.

(b) Now suppose, in place of the assumption $c \leqslant 0$, that $\sup_\Omega c^+ < \infty$, where $c^+ = \max\{c, 0\}$. Use the result of (a)(i) to deduce the following more general version of it: for each $\epsilon > 0$ and $\tau > 0$, there is $\delta = \delta(\epsilon, \tau) > 0$, with $\lim_{\tau \to 0^+} \delta(\epsilon, \tau) = \infty \;\; \forall \epsilon > 0$, such that if $0 \in \Omega \subset \{-d < x_1 < d\}$ and $\sup_\Omega c^+ \leqslant \tau$, then for any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f : \Omega \to \mathbb{R}$ satisfying $\Delta\, u + cu = f$ in Ω, we have

$$\sup_{\Omega \cap \{-d_1 < x_1 < d_1\}} |u| \leqslant (1 + \epsilon) \left(\sup_{\partial\, (\Omega \cap \{-d_1 < x_1 < d_1\})} |u| + (e^{2d_1} - 1) \sup_{\Omega \cap \{-d_1 < x_1 < d_1\}} |f| \right)$$

where $d_1 = \min\{d, \delta(\epsilon, \tau)\}$.

2

In this question we use the notation $B_\rho = \{x \in \mathbb{R}^n : \|x\| < \rho\}$ and $B = B_1$, where $\|\cdot\|$ denotes the Euclidean norm.

Let $\alpha \in (0,1)$ and $\beta > 0$. Let $b^1, b^2, \ldots, b^n, c \in C^{0,\alpha}(\overline{B})$ satisfy $\sum_{i=1}^n |b^i|_{0,\alpha;B} + |c|_{0,\alpha;B} \leqslant \beta$. Let L be the differential operator defined by $Lu = \Delta u + b^i D_i u + cu$, where Δ is the Laplacian.

(a) Show that for each $\delta \in (0,1)$, there is a constant $C = C(n, \alpha, \beta, \delta) \in (0, \infty)$ such that if $u \in C^{2,\alpha}(B)$ satisfies $Lu = 0$ in B, then

$$[D^2u]_{\alpha;\,B_{1/2}} \leqslant \delta [D^2 u]_{\alpha;\,B} + C|u|_{2;\,B}.$$

Explain *briefly* how to deduce from this estimate that there is a constant $C = C(n, \alpha, \beta)$ such that, if $u \in C^{2,\alpha}(B)$ satisfies $Lu = 0$ in B, then

$$|u|_{2,\alpha;B_{1/2}} \leqslant C|u|_{0;\,B}.$$

[*You are not required to give proofs of any additional results needed.*]

(b) Let $(u_k)_{k=1}^\infty$ be a sequence of functions in $C^2(B)$ satisfying $Lu_k = 0$ in B for every k. Suppose that there is a function $v : B \to \mathbb{R}$ such that $u_k \to v$ locally uniformly on B. Show that $v \in C^2(B)$ and that v satisfies $Lv = 0$ in B.

(c) Let u_k be as in (b), but instead of the hypothesis that $u_k \to v$ locally uniformly, assume that $v \in L^1(B)$ and that $u_k \to v$ in $L^1(B)$. Does the conclusion of (b) still hold? Give a proof if yes, or a counterexample if no.

(d) Let $(u_k)_{k=1}^\infty$ be a sequence of functions in $C^2(B) \cap C^0(\overline{B})$ satisfying $Lu_k = 0$ in B for every k. If there is a function $v \in C^2(\overline{B})$ such that $u_k \to v$ uniformly on ∂B, does it follow that a subsequence of (u_k) converges uniformly on B? Give a proof if yes, or a counterexample if no.

[In any part of the question, you may use without proof the following:

(i) Liouville's theorem for harmonic functions: *there does not exist a non-constant harmonic function w on $\mathbb{R}^n$ such that $[w]_{\alpha;\,\mathbb{R}^n} < \infty$ for some $\alpha \in (0,1)$.*

(ii) any existence theorem established in the course for solutions to the Dirichlet problem for elliptic operators.]

3 Let $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ be the open unit ball in $\mathbb{R}^n$, and let $\alpha \in (0,1)$, $\beta > 0$, $\lambda > 0$ be fixed constants. For $1 \leqslant i, j \leqslant n$, let a^{ij}, b^i be functions in $C^{0,\alpha}(\overline{B})$ satisfying $|a^{ij}|_{0,\alpha;B}, |b^i|_{0,\alpha;B} \leqslant \beta$ and $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for all $x \in B$, $\zeta \in \mathbb{R}^n$. Let $L : C^{2,\alpha}(\overline{B}) \to C^{0,\alpha}(\overline{B})$ be the linear differential operator given by $Lu = a^{ij}D_{ij}u + b^i D_i u$. Let $C_0^{2,\alpha}(\overline{B}) = \{u \in C^{2,\alpha}(\overline{B}) : u = 0 \text{ on } \partial B\}$.

(a) Suppose that functions $u \in C_0^{2,\alpha}(\overline{B})$ and $f \in C^{0,\alpha}(\overline{B})$ satisfy $Lu = f$ in B. Give the statements of: (i) an upper bound for $|u|_{0;B}$ in terms of an appropritate norm of f, and (ii) the global Schauder estimate satisfied by u.

(b) Given that the Laplacian $\Delta : C_0^{2,\alpha}(\overline{B}) \to C^{0,\alpha}(\overline{B})$ is a bijection, show that $L : C_0^{2,\alpha}(\overline{B}) \to C^{0,\alpha}(\overline{B})$ is a bijection.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a C^2 function with $g(0) = 0$, $g'(0) = 0$ and $\sup_{[-1,1]} |g''(t)| \leqslant 1$. Let $\mathcal{N} : C^{2,\alpha}(\overline{B}) \to C^{0,\alpha}(\overline{B})$ be defined by

$$\mathcal{N}u = Lu + g(u).$$

(i) For $u, u_1, u_2 \in C^{0,\alpha}(\overline{B})$ with $|u|_{0;B}, |u_1|_{0;B}, |u_2|_{0;B} \leqslant 1$, show that

$$|g(u)|_{0,\alpha;B} \leqslant |u|_{0,\alpha;B}^2$$

and that

$$|g(u_1) - g(u_2)|_{0,\alpha;B} \leqslant (|u_1|_{0,\alpha;B} + |u_2|_{0,\alpha;B})|u_1 - u_2|_{0,\alpha;B}.$$

(ii) Prove that there are constants $\epsilon_0 = \epsilon_0(n, \alpha, \beta, \lambda) \in (0,1)$ and $\delta_0 = \delta_0(n, \alpha, \beta, \lambda) > 0$ such that for each $c, f \in C^{0,\alpha}(\overline{B})$ with $|c|_{0,\alpha;B}, |f|_{0,\alpha;B} \leqslant \delta_0$, there exists a unique function $u \in C^{2,\alpha}(\overline{B})$ with $|u|_{2,\alpha;B} \leqslant \epsilon_0$ such that

$$\mathcal{N}u + cu = f \text{ in } B \text{ and } u = 0 \text{ on } \partial B.$$

4 In this question we use the notation $B_\rho = \{x \in \mathbb{R}^n : \|x\| < \rho\}$ and $B = B_1$, where $\|\cdot\|$ is the Euclidean norm.

For $1 \leqslant i, j \leqslant n$, let $a^{ij} \in L^\infty(B)$ be such that $\|a^{ij}\|_{L^\infty(B)} \leqslant \Lambda$ and $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for some constants $\Lambda \geqslant \lambda > 0$, all $\zeta \in \mathbb{R}^n$ and a.e. $x \in B$. Let $Lu = D_i\left(a^{ij}D_j u\right)$.

(a) Let $u \in W^{1,2}(B)$ be a non-negative weak supersolution of $Lu = 0$ in B. State without proof the *weak Harnack inequality* which gives a lower bound for $\inf_{B_{1/2}} u$.

(b) Let $u \in W^{1,2}(B)$ be a weak solution of $Lu = 0$ in B. Show that $u \in C^{0,\mu}(B_{1/8})$, and that

$$|u|_{0,\mu;B_{1/8}} \leqslant C\|u\|_{L^2(B)}$$

for some constants $\mu \in (0,1)$ and $C \in (0,\infty)$ depending only on n, λ and Λ. [*You may use without proof any other standard results established in the course provided they are stated clearly.*]

(c) If $u \in W^{1,2}(B)$ is a non-negative weak supersolution of $Lu = 0$ in B, prove that for each $\rho \in (0,1)$,

$$\int_{B_\rho} \frac{|Du|^2}{u^2} \leqslant C(1-\rho)^{-2}$$

where $C = C(n,\lambda,\Lambda) \in (0,\infty)$.

(d) For each $k = 1,2,3,\ldots$, let $u_k \in W^{1,2}(B)$ be a non-negative weak supersolution of $Lu = 0$ in B such that $0 < \sup_B u_k < \infty$. Prove that there is a subsequence $(u_{k'})$ and a non-negative function v such that $v \in W^{1,2}(B_\rho)$ for each $\rho \in (0,1)$, v is a weak supersolution of $Lu = 0$ in B and $\frac{u_{k'}}{(\sup_B u_{k'})} \to v$ in $L^2(B_\rho)$ and weakly in $W^{1,2}(B_\rho)$ for each $\rho \in (0,1)$.

[*Hint: recall the Rellich compactness theorem, which implies that a bounded sequence in $W^{1,2}(B_\rho)$ has a subsequence that converges in $L^2(B_\rho)$ and weakly in $W^{1,2}(B_\rho)$.*]

5 Throughout this question $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ is the open unit ball in $\mathbb{R}^n$, and $\varphi \in C^1(\overline{B})$ is a given function.

(a) For $1 \leqslant i, j \leqslant n$, let $\alpha^{ij} \in L^\infty(B)$ be such that $\alpha^{ij} = \alpha^{ji}$ for all i, j. Suppose that there is a constant $\gamma > 0$ such that $\gamma|\zeta|^2 \leqslant \alpha^{ij}(x)\zeta^i\zeta^j$ for a.e. $x \in B$ and all $\zeta \in \mathbb{R}^n$. Show that there is a unique function $u \in W^{1,2}(B)$ with $u - \varphi \in W_0^{1,2}(B)$ such that $D_i(\alpha^{ij} D_j u) = 0$ weakly in B.

Show further that if $\Gamma > 0$ is a constant such that $\|\alpha^{ij}\|_{L^\infty(B)} \leqslant \Gamma$ for all i, j, then $u \in C^{0,\beta}(\overline{B})$ for some $\beta = \beta(n, \gamma, \Gamma) \in (0, 1)$, and

$$\int_B |Du|^2\psi^2 \leqslant 4\left(\frac{\Gamma}{\gamma}\right)^2 \int_B u^2|D\psi|^2$$

for each $\psi \in C_c^1(B)$.

(b) For $1 \leqslant i, j \leqslant n$, let $a^{ij} \in C^0(\overline{B} \times \mathbb{R})$ be such that $a^{ij} = a^{ji}$ for all i, j, and suppose that there exist positive continuous functions $\lambda, \Lambda : \mathbb{R} \to \mathbb{R}$ such that $\lambda(t)|\zeta|^2 \leqslant a^{ij}(x,t)\zeta^i\zeta^j \leqslant \Lambda(t)|\zeta|^2$ for all $(x, t) \in \overline{B} \times \mathbb{R}$.

(i) Show that for any $v \in C^0(\overline{B})$, there exists $\mu = \mu(v) \in (0, 1)$ such that the linear Dirichlet problem

$$D_i(a^{ij}(x, v)D_j u) = 0 \;\text{ in } B, \;\; u = \varphi \;\text{ on } \partial B$$

has a unique weak solution $u \in W^{1,2}(B) \cap C^{0,\mu}(\overline{B})$.

(ii) State without proof the *Leray–Schauder fixed point theorem* for a continuous compact map T from a Banach space to itself.

(iii) Show that the quasilinear Dirichlet problem

$$D_i(a^{ij}(x, u)D_j u) = 0 \;\text{ in } B, \;\; u = \varphi \;\text{ on } \partial B$$

has a weak solution $u \in W^{1,2}(B) \cap C^0(\overline{B})$.

[In any part of the question, you may use without proof the following two results:

(i) the weak maximum principle for weak solutions: *if α^{ij} are as in (a) and $u \in W^{1,2}(B)$ solves $D_i(\alpha^{ij}D_j u) = 0$ weakly in B with $u - \varphi \in W_0^{1,2}(B)$, then $\sup_B |u| = \sup_{\partial B} |\varphi|$;*

(ii) global De Giorgi–Nash–Moser regularity theorem: *corresponding to any given constants $\gamma, \Gamma > 0$, there exist constants $\beta = \beta(n, \gamma, \Gamma) \in (0, 1)$ and $C = C(n, \gamma, \Gamma) \in (0, \infty)$ such that if the functions $\alpha^{ij} \in L^\infty(B)$ satisfy $\|\alpha^{ij}\|_{L^\infty(B)} \leqslant \Gamma$ for each i, j, and $\alpha^{ij}(x)\zeta^i\zeta^j \geqslant \gamma|\zeta|^2$ for a.e. $x \in B$, and if $u \in W^{1,2}(B)$ is a weak solution to $D_i\left(\alpha^{ij}D_j u\right) = 0$ in B with $u - \varphi \in W_0^{1,2}(B)$, then $u \in C^{0,\beta}(\overline{B})$ and satisfies $[u]_{\beta;\overline{B}} \leqslant C(|u|_{0;\overline{B}} + |D\varphi|_{0;\overline{B}})$.*]

END OF PAPER