UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2022 9:00 am to 12:00 pm

PAPER 106

FUNCTIONAL ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let A be a unital Banach algebra and let $x \in A$. Define the *spectrum of x in A* and show that it is a non-empty, compact subset of $\mathbb{C}$. Given a closed unital subalgebra B of A with $x \in B$, state and prove the most general form of the spectrum of x in B in terms of the spectrum of x in A. [No result about Banach algebras can be used without proof.]

Let X be the Banach space $\ell_1(\mathbb{Z}) = \{(x_n)_{n\in\mathbb{Z}} : \sum|x_n| < \infty\}$ equipped with the ℓ_1-norm $\|x\| = \sum|x_n|$. Let $T\colon X \to X$ be the bilateral shift operator given by $T\big((x_n)\big) = (y_n)$ where $y_n = x_{n-1}$ for all $n \in \mathbb{Z}$. Show that the spectrum of T in $\mathcal{B}(X)$ is contained in $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Let A be the smallest closed unital subalgebra of $\mathcal{B}(X)$ that contains T. Find the spectrum of T in A and in $\mathcal{B}(X)$. Justify your answers.

2

(a) State Goldstine's theorem. State and prove the Banach–Alaoglu theorem.

Let X be a Banach space and K be the closed unit ball B_{X^*} of X^* with the w^*-topology. Which of the following statements are true? Give a proof or a counterexample as appropriate.

1. X is separable $\implies$ K is w^*-sequentially compact
2. X is separable $\implies$ X^* is w^*-separable
3. X^* is w^*-separable $\implies$ X is separable

[*Results from general topology can be assumed without proof.*]

(b) A Banach space X is said to be *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in B_X$ satisfy $\|x - y\| \geqslant \varepsilon$, then $\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta$. Show that if $2 \leqslant p < \infty$, then $L_p = L_p[0,1]$ is uniformly convex. [*Hint: You may assume that*

$$\left|\frac{s+t}{2}\right|^p + \left|\frac{s-t}{2}\right|^p \leqslant \frac{|s|^p + |t|^p}{2}$$

holds for all real numbers s, t.]

Show that a uniformly convex Banach space X is reflexive. [*Hint: Fix $\varphi \in S_{X^{**}}$ and $\varepsilon > 0$. Consider a w^*-neighbourhood of φ of the form $\{\psi \in B_{X^{**}} : \psi(f) > 1 - \delta\}$ for suitable $f \in B_{X^*}$ and $\delta > 0$. It may also help to recall that $B_{X^{**}}$ is w^*-closed.*]

Deduce that if $1 < p < \infty$, then L_p is reflexive.

3

(a) Let X be a real Banach space. Define the *weak topology* on X.

Let $f\colon X \to \mathbb{R}$ be a linear map. Show that f is w-continuous if and only if $f \in X^*$.

Prove Mazur's theorem: a norm-closed convex subset K of X is weakly closed.

Show that X is reflexive if and only if B_X is weakly compact. Deduce that if X is reflexive, then X^* is reflexive as are Y and X/Y for any closed subspace Y of X.

(b) Let X be a real infinite-dimensional Banach space, and let (x_n) be a sequence in X that converges weakly to zero. Show that (x_n) is bounded.

Let K be the closed convex hull of $\{x_n : n \in \mathbb{N}\}$. Prove that K is weakly compact. [*Hint: You may assume that a weakly sequentially compact set is weakly compact. Then apply a diagonal argument to sequences of convex combinations of the* (x_i).]

By considering a suitable operator $X^* \to c_0$, show that K has empty interior in the norm topology. [*Hint: Argue by contradiction. You may assume without proof that every closed, infinite-dimensional subspace of* c_0 *contains a subspace isomorphic to* c_0.]

4

(a) Let T be a bounded linear map on a (non-zero) Banach space X. Assume that that the spectrum of T is contained in the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f\colon D \to \mathbb{C}$ be a holomorphic function with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that the series $\sum_{n=0}^{\infty} a_n T^n$ converges in the operator norm. [You may use any result from the course provided that you state it clearly.]

(b) Let T be a normal operator on the Hilbert space ℓ_2. If the spectrum $\sigma(T) = \{1\}$, does it follow that T equals the identity operator I? Does the answer change if T is not assumed normal? Justify your answers. [You may assume results about Banach algebras, but no result specific to the spectral theory of C^*-algebras can be used without proof.]

(c) State and prove the Gelfand-Naimark theorem. Define what it means for an element x of a C^*-algebra A to be *positive*. Show that if x is a positive element of a C^*-algebra A, then there is a positive element y of A such that $y^2 = x$. [You may assume results about Banach algebras, but no result specific to the spectral theory of C^*-algebras can be used without proof.]

5

(a) Show that if C is an open, convex subset of a locally convex space X with $0 \in C$ and $x_0 \in X \setminus C$, then there exists $f \in X^*$ such that $f(x_0) > f(x)$ for all $x \in C$. Deduce, or otherwise show, that if x_0 is a non-zero element of a Banach space X, then there exists $f \in S_{X^*}$ with $f(x_0) = \|x_0\|$. Deduce that if Y is a closed subspace of a Banach space X and $x_0 \in X \setminus Y$, then there exists $f \in S_{X^*}$ such that $Y \subset \ker f$ and $f(x_0) = d(x_0, Y)$.

Let X be a Banach space, F be a finite-dimensional subspace of X^* and $\varphi \in B_{X^{**}}$. Show that for all $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| < 1 + \varepsilon$ and $f(x) = \varphi(f)$ for all $f \in F$. Deduce Goldstine's theorem.

[*The Hahn–Banach extension theorem for real vector spaces and positive homogeneous, subadditive functionals can be used without proof.*]

(b) Let X be a Banach space. A subspace Z of X^* is said to be *norming* for X if there exists a constant $c > 0$ such that

$$c\|x\| \leqslant \sup_{g \in B_Z} |g(x)|$$

for all $x \in X$. If this holds, we say Z is c-*norming* for X.

Assume X is not reflexive and $\varphi \in B_{X^{**}} \setminus X$. Show that $\ker\varphi$ is c-norming for X where $c = \frac{d}{d+1}$ and $d = d(\varphi, X)$. [*Hint: first show that* $d(x, \mathrm{span}\{\varphi\}) \geqslant c$ *whenever* $x \in X$ *and* $\|x\| = 1$.]

END OF PAPER