UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2022 9:00 am to 12:00 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **QUESTION 1** and no more than **TWO** other questions.
There are **FOUR** questions in total.
Question 1 is worth 40% of the total marks. Questions 2, 3 and 4 carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 *You should attempt **all parts** of this question.*

a) Consider the heat equation for $(t, x, y) \in \mathbb{R}^3$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

with the following data:

$$u|_{\Sigma} = u_0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\Sigma} = u_1,$$

where $\Sigma \subset \mathbb{R}^3$ is a real analytic hypersurface with unit normal ν, and $u_0, u_1 : \Sigma \to \mathbb{R}$ are real analytic. Find a condition on Σ such that a real analytic solution to this problem exists in a neighbourhood of Σ, clearly identifying any theorems you apply.

b) State the Lax–Milgram theorem. Explain briefly how the theorem may be used to find solutions to the following PDE problem:

$$\begin{cases} -\Delta u + u_{x_n} + u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

where $U \subset \mathbb{R}^n$ is an open, bounded set and $f \in L^2(U)$ is given.
You may state without justification the weak formulation of this problem.

c) Let $U, V \subset \mathbb{R}^n$ be open with $V \subset\subset U$. For $u : U \to \mathbb{R}$, define the i^{th} difference quotient $\Delta_i^h u(x) := h^{-1}\left[u(x + he_i) - u(x)\right]$ for $x \in V$, $0 < |h| < \text{dist}\,(V, \partial U)$, $i = 1, \ldots, n$.

i) Show that if $u \in H^1(U)$, then

$$\left\|\Delta_i^h u\right\|_{L^2(V)} \leqslant \|D_i u\|_{L^2(U)}, \qquad \text{for all } 0 < |h| < \frac{1}{2}\text{dist}\,(V, \partial U),\ i = 1, \ldots, n.$$

ii) Suppose $u \in L^2(U)$ satisfies

$$\left\|\Delta_i^h u\right\|_{L^2(V)} \leqslant C, \qquad \text{for all } 0 < |h| < \frac{1}{2}\text{dist}\,(V, \partial U),\ i = 1, \ldots, n.$$

Show that $u \in H^1(V)$ with $\|D_i u\|_{L^2(V)} \leqslant C$ for each $i = 1, \ldots, n$.

d) State the Rellich–Kondrachov theorem for $H^1(U)$, where $U \subset \mathbb{R}^n$ is open and bounded, with smooth boundary. Give an example to show that the assumption that U be bounded is necessary.

[40]

2 *You should attempt* ***at most two*** *from questions 2, 3, 4.*

Let $U \subset \mathbb{R}^n$ be open, connected, and bounded, with smooth boundary. Let

$$H = \left\{ u \in H^1(U) : \int_U u(x)dx = 0 \right\}.$$

a) Show that H is a Hilbert space when equipped with the standard H^1-inner product.

b) Show that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(U)} \leqslant C\|Du\|_{L^2(U)}, \qquad \text{for all } u \in H.$$

You may assume the Rellich–Kondrachov theorem.
[Hint: assume for contradiction that the result is false, and consider a sequence $(u_i)_{i=1}^{\infty}$ with $u_i \in H$ satisfying $\|u_i\|_{L^2} > i\|Du_i\|_{L^2}$.]

c) Suppose that there exist $w \in H$ and $\gamma > 0$ such that:

$$\|u\|_{L^2(U)} \leqslant \gamma^{-\frac{1}{2}}\|Du\|_{L^2(U)}, \qquad \text{for all } u \in H,$$

with equality for $u = w$. By considering $u = w + tv$ for $t \in \mathbb{R}$, $v \in H$, show that

$$0 \leqslant 2t\left[(Dw, Dv)_{L^2} - \gamma(w, v)_{L^2}\right] + t^2\left[\|Dv\|^2_{L^2(U)} - \gamma\|v\|^2_{L^2(U)}\right].$$

Deduce that w is the weak solution to the PDE problem:

$$\begin{cases} -\Delta w = \gamma w & \text{in } U, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial U, \\ \int_U w(x)dx = 0. & \end{cases}$$

[30]

3 *You should attempt* ***at most two*** *from questions 2, 3, 4.*

a) Prove the Lax–Milgram theorem for a bilinear form B defined on a real Hilbert space H.

b) Let $U \subset \mathbb{R}^n$ be open and bounded with C^∞ boundary, and let $A : \mathbb{R}^n \to \mathrm{Mat}(2 \times 2)$ be a real-valued matrix whose components are $C^\infty(\bar{U})$. Consider the system of elliptic equations

$$-\Delta\phi + A\phi = F \quad \text{in } U\,, \tag{1}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^2$ are the unknowns, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplacian acting on each component of ϕ, and $F : \mathbb{R}^n \to \mathbb{R}^2$ is given. We suppose that ϕ is subject to the boundary conditions

$$\phi = \begin{pmatrix} 0 & 0 \end{pmatrix}^T \quad \text{on } \partial U\,. \tag{2}$$

(i) Define a *weak solution* for the equations (1) subject to the boundary conditions (2), clearly identifying the space to which ϕ belongs.

Hint: it may be useful to use the notation $v \in V^2$ to denote a vector $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ whose components v_1, v_2 lie in a space V.

(ii) Show that if a weak solution is such that each component of ϕ is in $C^2(\bar{U})$, then the equations (1), (2) hold classically.

(iii) The *formal adjoint* $P^\dagger$ of a linear differential operator P is defined to satisfy

$$(Pu, v)_{L^2(U)} = (u, P^\dagger v)_{L^2(U)} \quad \forall\, u, v \in C_c^\infty(U)\,.$$

Compute the formal adjoint for the operator $L = -\Delta + A$ appearing in (1) and show that $L = L^\dagger$ if and only if the matrix A is symmetric.

(iv) Suppose that A is a positive semi-definite matrix. Show that the equations (1), (2) admit a unique weak solution for any F in an appropriate space that you should specify. [30]

4 *You should attempt **at most two** from questions 2, 3, 4.*

Let $U \subset \mathbb{R}^3$ be open, and bounded with smooth boundary, and let $T > 0$ be fixed. Define $U_T := (0,T) \times U$, $\Sigma_t := \{t\} \times U$, $\partial^* U_T := [0,T] \times \partial U$. Given $\psi \in L^2(U)$ and $f \in L^2(U_T)$, a weak solution to the linear heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f & \text{in } U_T \\ u = 0 & \text{on } \partial^* U_T \\ u = \psi & \text{on } \Sigma_0 \end{cases} \qquad (\diamond)$$

is a function $u \in L^2((0,T); H^1_0(U))$ such that

$$\int_{U_T} (-uv_t + Du \cdot Dv) dxdt = \int_{\Sigma_0} \psi v dx + \int_{U_T} f v dxdt$$

holds for all $v \in H^1(U_T)$ with $v = 0$ on Σ_T. You may assume that for any $\psi \in H^1_0(U)$ and $f \in L^2(U_T)$, a unique weak solution exists satisfying

$$\|u\|_{L^\infty_t H^1_x} := \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t,\cdot)\|_{H^1(U)} \leqslant \alpha(\|\psi\|_{H^1(U)} + \|f\|_{L^2(U_T)}),$$

for some constant α depending only on U, T.

a) Let $w \in L^\infty((0,T); H^1_0(U))$. Show that $w^3 \in L^2(U_T)$ with

$$\|w^3\|_{L^2(U_T)} \leqslant \beta T^{\frac{1}{2}} \|w\|^3_{L^\infty_t H^1_x}$$

for some $\beta > 0$ depending only on U. If, further, $\tilde{w} \in L^\infty((0,T); H^1_0(U))$, show that

$$\|w^3 - \tilde{w}^3\|_{L^2(U_T)} \leqslant \gamma T^{\frac{1}{2}} \|w - \tilde{w}\|_{L^\infty_t H^1_x} (\|w\|^2_{L^\infty_t H^1_x} + \|\tilde{w}\|^2_{L^\infty_t H^1_x})$$

for some $\gamma > 0$ depending only on U.

b) Fix $\psi \in H^1_0(U)$. Let $X_{b,\tau} = \{u \in L^\infty((0,\tau); H^1_0(U)) : \|u\|_{L^\infty_t H^1_x} \leqslant b\}$. Let A be the map which takes $w \in X_{b,\tau}$ to the unique weak solution in $L^\infty((0,\tau); H^1_0(U))$ of $(\diamond)$ with f given by $f = -w^3$. Show that $A : X_{b,\tau} \to X_{b,\tau}$ is a contraction map provided $b > 0$ is sufficiently large and $0 < \tau < T$ is sufficiently small.

c) Deduce that the nonlinear heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - u^3 & \text{in } U_\tau \\ u = 0 & \text{on } \partial^* U_\tau \\ u = \psi & \text{on } \Sigma_0 \end{cases}$$

has a weak solution $u \in L^\infty((0,\tau); H^1_0(U))$, provided τ is sufficiently small.

[30]

END OF PAPER