UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 9:00 am to 12:00 pm

PAPER 104

INFINITE GROUPS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Throughout this question, G and H are nontrivial groups, given by presentations $\langle X \mid R\rangle$ and $\langle Y \mid S\rangle$, respectively. The sets X and Y are assumed to be disjoint.

(a) Write down a presentation for the *free product* $G * H$ of G and H. [2]

(b) Define the notion of a *reduced word* in G and H. By constructing a suitable action of $F(X \sqcup Y)$ on the set of all reduced words in G and H, prove that any two distinct reduced words in G and H represent distinct elements of $G * H$. [9]

(c) Prove that $G*H$ has trivial centre. [You may assume without proof that every element of $G * H$ is represented by a reduced word in G and H.] [6]

(d) Suppose that G and H both have order at least 3. Prove that $G * H$ has a subgroup isomorphic to a free group of rank 2. [You may use the Universal Property of free groups without proof.] [8]

2 (a) Give, with proof, an example of each of the following:

1. A finitely generated nilpotent nonabelian group; [5]
2. A nilpotent group which is not polycyclic; [3]
3. A finitely generated soluble group which is not polycyclic. [7]

(b) Let:

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Let:

$$L = \left\{ \begin{pmatrix} w & x & a_{1,3} & a_{1,4} \\ y & z & a_{2,3} & a_{2,4} \\ 0 & 0 & 1 & a_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Z} \text{ and } \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \langle A \rangle \right\}$$

Prove that L is polycyclic but not virtually nilpotent. [10]

[Throughout this Question you may use without proof any standard results about the class of nilpotent, polycyclic or soluble groups, provided you state them clearly.]

3 Let T_3 be the *ternary rooted tree*, that is, T_3 is the graph (V, E) with vertex-set $V = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}^*$ (the set of finite formal words in the alphabet $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$) and edge-set $E = \big\{(v, v\epsilon) : v \in V, \epsilon \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}\big\}$. Let $\mathrm{Aut}(T_3)$ be the group of graph-automorphisms of T_3.

(a) Explain briefly why the set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \subset V$ is invariant under the action of $\mathrm{Aut}(T_3)$ on V. [2]

Define $a, b \in \mathrm{Aut}(T_3)$ by:

$$a(\mathbf{0} \cdot v) = \mathbf{1} \cdot v; \qquad a(\mathbf{1} \cdot v) = \mathbf{2} \cdot v; \qquad a(\mathbf{2} \cdot v) = \mathbf{0} \cdot v;$$
$$b(\mathbf{0} \cdot v) = \mathbf{0} \cdot a(v); \qquad b(\mathbf{1} \cdot v) = \mathbf{1} \cdot v; \qquad b(\mathbf{2} \cdot v) = \mathbf{2} \cdot b(v).$$

(b) Prove that a and b have order 3. [5]

For the remainder of the question G is the subgroup of $\mathrm{Aut}(T_3)$ generated by the set $S = \{a, b\}$.

(c) Let $\mathrm{Stab}_G(1)$ be the kernel of the action of G on $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. Prove that $\{b\}$ is a normal generating set for $\mathrm{Stab}_G(1)$ and prove that $\{b, aba^{-1}, a^{-1}ba\}$ is a generating set for $\mathrm{Stab}_G(1)$. [6]

(d) Show that there is an injective homomorphism:

$$\phi : \mathrm{Stab}_G(1) \to \mathrm{Aut}(T_3) \times \mathrm{Aut}(T_3) \times \mathrm{Aut}(T_3)$$

given by $\phi(g) = (g_{\mathbf{0}}, g_{\mathbf{1}}, g_{\mathbf{2}})$, where $g(\mathbf{i} \cdot v) = \mathbf{i} \cdot g_{\mathbf{i}}(v)$ for $\mathbf{i} = \mathbf{0}, \mathbf{1}$ or $\mathbf{2}$. Prove that the image of ϕ is contained in $G \times G \times G$ and show that G is an infinite group. [6]

(e) Let K be the normal closure in G of $x = [a, b]$. Prove that K has finite index in G. By considering the element $[a^{-1}ba, x]$, or otherwise, prove that K contains a subgroup isomorphic to $K \times K \times K$. [6]

4 (a) Define what it means for a group to be *residually finite.* [3]

(b) Prove that every finitely generated abelian group is residually finite. [6]

(c) Prove that for any finitely generated group G and any finite group Q, there are finitely many homomorphisms from G to Q. [2]

The group H is called *Hopfian* if every surjective homomorphism $H \to H$ is bijective.

(d) Using (c), prove that every finitely generated residually finite group is Hopfian. [6]

(e) Let X be a finite set and let $F(X)$ be the free group on X. Let $Y \subset F(X)$ be a generating set for $F(X)$, with $|Y| = |X|$. Prove that Y is a basis for $F(X)$. [You may assume without proof that $F(X)$ is residually finite. You may assume any other standard properties of free groups, provided you state them clearly.] [5]

(f) Prove that $F(\mathbb{N})$ (the free group on the set of natural numbers) is not Hopfian. [3]

5 Let G and H be groups.

(a) Define the (regular, restricted) *wreath product* $H \operatorname{wr} G$ of H by G. Prove that if G and H are finitely generated, then so is $H \operatorname{wr} G$. [8]

(b) Prove that, if G is infinite and H is nonabelian, then $H \operatorname{wr} G$ is not residually finite. [6]

(c) Prove that, if H is a nontrivial finitely generated group, then $H \operatorname{wr} \mathbb{Z}$ has exponential growth. [6]

(d) Prove that, if H is a nontrivial group, then $H \operatorname{wr} \mathbb{Z}$ has trivial centre. [5]

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