UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2022 9:00 am to 12:00 pm

PAPER 102

FINITE DIMENSIONAL LIE AND ASSOCIATIVE ALGEBRAS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let L be a finite dimensional complex Lie algebra.

For any Lie subalgebra L_1 of L, let $I(L_1) = \{x \in L : [x, L_1] \leqslant L_1\}$.

For $y \in L$, let $L_{0,y} = \{x \in L : (\mathrm{ad}(y))^m(x) = 0 \text{ for some } m\}$, the generalised 0-eigenspace of $\mathrm{ad}(y)$. Show that $L_{0,y}$ is a Lie subalgebra of L, and that $I(L_{0,y}) = L_{0,y}$.

Let L_1 be any Lie subalgebra containing $L_{0,y}$. Show that $I(L_1) = L_1$.

What does it mean for L to be nilpotent?

Show that if L is nilpotent and L_1 is any proper subalgebra of L then $I(L_1)$ strictly contains L_1.

Show that if $L_{0,y} = L$ for all $y \in L$ then L is a nilpotent Lie algebra. [You should prove any results that you use.]

2

(a) What is a finite root system Φ? What does it mean for the root system to be reduced? What is the Weyl group $W(\Phi)$ of the root system? What is a base Δ of the root system Φ?

Define the co-root (or inverse root) of a root $\alpha \in \Phi$. You may assume that the set of co-roots form a root system Φ'. Let Δ be a base of Φ. Show that the co-roots of the elements of Δ form a base of Φ'. [You may assume any properties of $W(\Phi)$ that you need.]

(b) Let L be a finite dimensional complex Lie algebra.

What does it mean for a Lie subalgebra H of L to be a Cartan subalgebra?

Explain briefly, without proofs, how to obtain a finite root system from a semisimple finite dimensional Lie algebra L.

[You may assume that every Cartan subalgebra H of L is a maximal abelian subalgebra all of whose elements are semisimple, and that the Killing form on L restricts to give a non-degenerate symmetric bilinear form on H.]

3

(a) Let L be a finite dimensional complex Lie algebra.

What does it mean for L to be semisimple?

Prove Weyl's theorem about the complete reducibility of finite dimensional representations of a semisimple finite dimensional complex Lie algebra. [You may assume commuting properties of a Casimir element.]

Choose a subalgebra L of sl_3 isomorphic to sl_2. Let θ be the adjoint representation of sl_3, and consider the restriction of θ to your chosen copy of sl_2. Demonstrate that this representation of sl_2 is completely reducible.

(b) Let R be a finite dimensional associative algebra.

Suppose M is a finite dimensional completely reducible right R-module. What is the structure of $End_R(M)$? [You should prove any results that you use.]

4

Let Q be a quiver with finitely many vertices and finitely many arrows. Let k be an infinite field.

What does it mean for a vertex to be a sink?

What is a representation of Q over the field k? What is a morphism of representations?

What does it mean for Q to be of finite representation type over the field k?

Suppose that vertex i of quiver Q is a sink, and obtain a new quiver Q' by reversing all the arrows which have that vertex as target. Show that Q' is of finite representation type if and only if Q is of finite representation type.

Let Q_1 be the quiver with vertices $1, 2, 3, 4$ and 5 with one arrow from each of $1, 2, 3$ and 4 to 5. (There are four arrows in total.)

Show that Q_1 is not of finite representation type.

Deduce that any quiver obtained from Q_1 by reversing some of the arrows is also not of finite representation type.

END OF PAPER