UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2022 1:30 pm to 4:30 pm

PAPER 101

COMMUTATIVE ALGEBRA

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) If $(A, \mathfrak{m})$ is a Noetherian local ring, give the definition of an *ideal of definition* of A.

For a finitely generated A-module M and ideal of definition I, state the relationship between the Hilbert function

$$\chi(M, I; n) = \ell(M/I^n M)$$

and $\dim M$.

(b) Let k be a field. Suppose $f \in (x, y) \subseteq k[x, y]$ is a polynomial. Let

$$A = (k[x, y]/(f))_{(x,y)}.$$

Calculate the Hilbert function $\chi(A, \mathfrak{m}; n)$ of A.

Calculate the Hilbert polynomial of A, i.e., the polynomial function in n which agrees with $\chi(A, \mathfrak{m}; n)$ for large n.

What is a simple way of describing the coefficient of the leading term of the Hilbert polynomial in terms of f?

(c) Let k be a field, and consider the graded ring $S = k[x, y]$ where the degree of x is 1 and the degree of y is 2, so that, e.g., $x^2y + y^2$ is a homogeneous element of S of degree 4. Calculate the Hilbert function

$$F_S(n) = \ell(S_n).$$

Does this agree with a polynomial for large n? If not, why does this not contradict a result from lectures?

2

Let $p \in \mathbb{Z}$ be a prime number, and denote by $\mathbb{Z}_p$ the completion of $\mathbb{Z}$ at the prime ideal (p).

(a) Give a brief description of $\mathbb{Z}_p$.

(b) For $a \in \mathbb{Z}_p$, denote $\operatorname{ord}(a) = \sup\{m \mid a \in (p^m)\}$ (so that $\operatorname{ord}(p^2) = 2$ and $\operatorname{ord}(0) = +\infty$). Set

$$\mathbb{Z}_p\langle T\rangle = \left\{ \sum_{n=0}^{\infty} a_n T^n \mid a_n \in \mathbb{Z}_p, \operatorname{ord}(a_n) \to +\infty \text{ when } n \to +\infty \right\}.$$

Note $\mathbb{Z}[T] \subseteq \mathbb{Z}_p\langle T\rangle \subset \mathbb{Z}_p[[T]]$ as sets, where the latter is the ring of formal power series in T with coefficients in $\mathbb{Z}_p$.

Show that $\mathbb{Z}_p\langle T\rangle$ is a subring of $\mathbb{Z}_p[[T]]$.

Show that $\mathbb{Z}_p\langle T\rangle$ is the completion of $\mathbb{Z}[T]$ with respect to the ideal $(p) \subseteq \mathbb{Z}[T]$.

3

Let S be a Noetherian graded ring. If $I \subseteq S$ is an arbitrary ideal, denote by I^* the ideal contained in I generated by all homogeneous elements of I.

(a) If $\mathfrak{p}$ is prime, show that $\mathfrak{p}^*$ is also prime.

(b) If $\mathfrak{p}$ is a homogeneous prime ideal and $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal, show that $\mathfrak{q}^*$ is also $\mathfrak{p}$-primary.

(c) If $\mathfrak{p}$ is an inhomogenenous prime ideal, show that there are no primes contained between $\mathfrak{p}$ and $\mathfrak{p}^*$. Show that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + 1$. [Hint: it may be useful to consider the following construction. Let R be a graded domain, and $U \subseteq R$ the set of non-zero homogeneous elements. Consider the ring $U^{-1}R$.]

4

Let A be a Noetherian domain, M an A-module. We say M is *torsion-free* if $\mathrm{Ann}(m) = 0$ for all $m \in M$. We denote by M^* the A-module $\mathrm{Hom}_A(M, A)$.

Define a natural map $M \to M^{**}$, by $a \mapsto (f \mapsto f(a))$. We say M is *reflexive* if it is finitely generated as an A-module and the natural map $M \to M^{**}$ is an isomorphism.

(a) Prove that if M is a finitely generated A-module, then there is an exact sequence

$$0 \longrightarrow M^* \longrightarrow N \longrightarrow P \to 0$$

where N is a finitely generated free A-module and P is torsion-free.

(b) Prove that an A-module M is reflexive if and only if it can be included in an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \to 0$$

where N is a finitely generated free A-module and P is torsion-free. Thus M^* is reflexive whenever M is a finitely generated A-module. [Hint: You might find it useful to localize at the zero ideal of A in the course of the proof.]

5

Consider the ideal

$$I = (x^3y^2,\, x^2y^3 - x^2yzw,\, x^2y^2z - x^2z^2w,\, x^3yz,\, x^3z^2) \subseteq k[x, y, z, w],$$

where k is a field.

(a) Show that the isolated associated primes of $k[x, y, z, w]/I$ are

$$\mathfrak{p}_1 = (x), \qquad \mathfrak{p}_2 = (y, z).$$

Find the radical of I.

Given a primary decomposition $I = \bigcap_i \mathfrak{q}_i$ of I, give generators for the ideals $\mathfrak{q}_1, \mathfrak{q}_2$ with $\sqrt{\mathfrak{q}_1} = \mathfrak{p}_1$, $\sqrt{\mathfrak{q}_2} = \mathfrak{p}_2$.

(b) Show that $\mathfrak{p}_3 = (x, y^2 - zw)$ is also an associated prime of $k[x, y, z, w]/I$. Show that $\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ is the complete set of associated primes, by finding a primary decomposition for I or otherwise. [Note: You do not need to show $\mathfrak{p}_3$ is prime.]

6

Let A be a ring, $S \subseteq A$ a multiplicatively closed subset.

(a) Show that giving an $S^{-1}A$-module M is the same as giving an A-module M such that for any $s \in S$, the A-module endomorphism $m \mapsto s \cdot m$ of M is an automorphism.

Show that if M is an A-module such that multiplication by each element of S acts as an automorphism of M, then the natural map $M \to S^{-1}M$ is an isomorphism.

(b) Let M be an A-module and $N' \subseteq S^{-1}M$ an $S^{-1}A$-submodule. Let N be the inverse image of N' under the natural map $M \to S^{-1}M$. Show that $N' \cong S^{-1}N$.

END OF PAPER