UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2021 12:00 pm to 2:00 pm

PAPER 354

GAUGE/GRAVITY DUALITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **TWO** questions in total.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider the vacuum 4-point function of scalar primary operators $\mathcal{O}$ in a CFT on a flat spacetime $\mathcal{M}$ with $d > 2$ dimensions:

$$F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle 0|\mathcal{O}(\mathbf{x}_1)\mathcal{O}(\mathbf{x}_2)\mathcal{O}(\mathbf{x}_3)\mathcal{O}(\mathbf{x}_4)|0\rangle,$$

whose domain $\mathcal{D}$ consists of those points in $\mathcal{M}^4 := \mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M}$ for which no two points coincide (i.e. $\mathbf{x}_i \neq \mathbf{x}_j$ when $i \neq j$). Fortunately you will not actually need to calculate any 4-point functions for this exam, but pretend for a moment that you were planning to do such a calculation. To make your life easier, you would wish to exploit conformal symmetry to the maximum degree possible.

What is the minimal dimension of a subdomain $\mathcal{S} \subseteq \mathcal{D}$, such that after calculating F in $\mathcal{S}$, conformal symmetry suffices to determine F for all other values in $\mathcal{D}$? Justify your answer. [10]

[Hint: Since the 4-point function is analytic in position space, you may optionally choose to analyze this problem in Euclidean signature, which will not change the answer. You might also find it useful to extend the spacetime $\mathcal{M}$ to a conformally completed spacetime $\overline{\mathcal{M}}$ with one or more additional points.]

(b) Consider the case of a holographic CFT, living in 2+1 dimensional Minkowksi spacetime. Let $\mathcal{O}(\mathbf{x})$ be a single-trace primary scalar operator with conformal dimension $\Delta_{\mathcal{O}} = 2$. Let the corresponding CFT source be $J(\mathbf{x})$, and let the dual bulk field be $\phi(\mathbf{x}, z)$.

Working in the limit where the bulk theory is free, use a Weyl transformation to map AdS to a flat spacetime, and find the following:

(i) The bulk equations of motion for ϕ; [5]

(ii) The holographic dictionary relating $\mathcal{O}$ and J to bulk quantities. [5]

Now suppose that there is a nonzero source J. It is possible to calculate its effect on the bulk field ϕ, by using the bulk-boundary retarded Green's function $G(\mathbf{x}, z|\mathbf{x}')$, which is defined to satisfy the relation:

$$\phi(\mathbf{x}, z) = \int d^3\mathbf{x}'\, J(\mathbf{x}')G(\mathbf{x}, z|\mathbf{x}') + \phi_0(\mathbf{x}, z),$$

where ϕ_0 is the solution obtained by evolving $\phi(t = -\infty)$ without any sources.

(iii) Calculate the value of $G(\mathbf{x}, z) \equiv G(\mathbf{x}, z|0)$. In other words, find the value of $\phi(\mathbf{x}, z)$ in the bulk when

$$J = \delta^3(\mathbf{x}),$$

and the CFT starts off in the vacuum state. *[Hint: use the method of images.]* [10]

2 Consider a holographic CFT, defined in 2+1 dimensional Minkowski spacetime, in the limit where the bulk geometry is described by *classical* general relativity. The vacuum state of this theory is dual to a 3+1 dimensional AdS-Poincaré geometry, whose 4-dimensional Newton's constant is G. We may take the coordinates of the dual bulk geometry to be:

$$ds^2 = \frac{1}{z^2}(-dt^2 + dx^2 + dy^2 + dz^2) \qquad z > 0,$$

setting the AdS radius to 1 for simplicity.

In this problem you will be analyzing the holographic entanglement of the vacuum state, for two different regions of the boundary CFT, namely a disk and a strip. [You may assume without proof the uniqueness of a candidate holographic entropy surface.]

(a) The first region is a disk D with radius R, e.g. the region $r \equiv \sqrt{x^2 + y^2} \leqslant R$ on the $t = 0$ Cauchy slice.

(i) Find the holographic entanglement surface γ_D in the bulk geometry. [5]

[Hint: the boundary of D is invariant under a conformal inversion $\mathbb{Z}_2$ symmetry: $x \to xR^2/r^2$, $y \to yR^2/r^2$. How does this symmetry extend to the bulk geometry?]

(ii) Calculate the entropy S of the disk D. Make sure to suitably regulate any ultraviolet divergence that arises, so that the resulting expression depends explicitly on an ultraviolet cutoff of length ϵ. [8]

(b) The second region is a strip T of width $2R$, e.g. $|x| \leqslant R$ on the $t = 0$ Cauchy slice.

[Note: Since this region is invariant under translations in the y-direction, it is not really meaningful to calculate its total entropy S (which is infinite), but rather the entropy *per unit length* L. In what follows, you will not be asked to find an explicit expression for S/L since solving the differential equations involved is difficult.]

(i) Write down an action for $z(x)$, where z is the position of the holographic entropy surface γ_T at a given value of x. [4]

(ii) Using the existence of a conserved quantity associated with a symmetry of your action, derive the following first order differential equation for the holographic entanglement surface γ_T:

$$\frac{dz}{dx} = \pm\sqrt{\frac{z_{\max}^4}{z^4} - 1},$$

where $z_{\max}$ is the maximum value of the z coordinate attained by the solution $z(x)$. [8]

(c) Using your results from the previous parts—or otherwise—determine which of the two holographic entropy surfaces lies *deeper* in the bulk. In other words, if we compare the surfaces γ_D and γ_T, which of them achieves the greatest value of the z coordinate? Justify your answer. [5]

END OF PAPER