UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 14 June, 2021 12:00 pm to 2:00 pm

PAPER 352

NON-NEWTONIAN FLUID MECHANICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **TWO** questions in total.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A Generalised Newtonian fluid of viscosity $\eta(\dot\gamma)$, where $\dot\gamma$ is the magnitude of shear rate, is located inside a stationary cylindrical pipe of radius R and length L. A pressure drop Δp is applied between the two ends of the pipe and drives a flow. All inertial effects in the fluid are neglected.

(a) Explain how the magnitude of shear rate $\dot\gamma$ is defined mathematically.

(b) Give two examples of rheological phenomena that will not be captured by a Generalised Newtonian fluid model.

(c) Show that the value of the shear stress at the wall, denoted by τ_R, can be computed using an overall force balance on the fluid. Deduce that the value of τ_R is set by the pipe geometry and the applied pressure. Determine the shear stress τ_{rz} throughout the pipe.

(d) First assume that the fluid obeys the power-law constitutive relationship $\eta(\dot\gamma) = K\dot\gamma^{n-1}$. Calculate the shear rate throughout the pipe. Integrate to obtain the velocity profile in the pipe.

(e) Now assume that the fluid follows an unknown constitutive relationship, $\eta(\dot\gamma)$, which you wish to estimate. Show that the flow rate in the pipe, Q, may be evaluated as $Q = \pi\int_0^R \dot\gamma(r) r^2 \mathrm{d}r$. Use a change of variable in the integral to write Q as an integral of the shear stress between $\tau_{rz} = 0$ and $\tau_{rz} = \tau_R$. Show that the value of the shear rate at the wall, $\dot\gamma_R$, may be obtained as a derivative of the product $Q \times \tau_R^n$, where n is some power to be determined. Deduce the value of the viscosity at the wall, $\eta(\dot\gamma_R)$, as a similar expression.

Hint: The inertialess Cauchy equations in cylindrical coordinates are

$$
\begin{aligned}
\frac{\partial p}{\partial r} &= \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z},\\
\frac{1}{r}\frac{\partial p}{\partial \theta} &= \frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z},\\
\frac{\partial p}{\partial z} &= \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}.
\end{aligned}
$$

2 A Giesekus fluid has a constitutive relationship between the deviatoric stress tensor, $\boldsymbol{\tau}$, and the shear rate tensor, $\dot{\boldsymbol{\gamma}}$, given by

$$\boldsymbol{\tau} + \lambda \overset{\triangledown}{\boldsymbol{\tau}} + \alpha \frac{\lambda}{\eta} \boldsymbol{\tau} \cdot \boldsymbol{\tau} = \eta \dot{\boldsymbol{\gamma}}, \qquad (\dagger)$$

where λ, α and η are positive constants.

(a) In Eq. (†), explain the physical meaning of the constants λ and η. What are the dimensions of α? Explain the meaning of the symbol $\overset{\triangledown}{\boldsymbol{\tau}}$ and its origin.

(b) What is the type of fluid obtained when $\alpha = 0$? Describe qualitatively the behaviour of that fluid in steady extension. Propose an experiment to estimate the value of λ.

(c) We next consider a Giesekus fluid with $\alpha > 0$ in a two-dimensional steady shear flow in the xy plane ($\mathbf{u} = \dot{\gamma} y \mathbf{e}_x$, $\dot{\gamma} > 0$). Assuming a symmetric second-rank tensor for $\boldsymbol{\tau}$ with $\tau_{xz} = \tau_{yz} = 0$, compute all components of $\overset{\triangledown}{\boldsymbol{\tau}}$ and $\boldsymbol{\tau} \cdot \boldsymbol{\tau}$ and deduce the four component equations resulting from Eq. (†).

(d) The value of α is assumed to be small so we solve the problem asymptotically as a power expansion, i.e. $\boldsymbol{\tau} = \boldsymbol{\tau}^0 + \alpha \boldsymbol{\tau}^1 + \mathcal{O}(\alpha^2)$. Compute the components of the stress at order zero, $\boldsymbol{\tau}^0$. Use that result to compute the stress components at order one, $\boldsymbol{\tau}^1$.

How does the viscosity vary with the shear rate when $\alpha > 0$? For what range of shear rates is this asymptotic solution expected to be valid?

Determine the leading-order values in α of the two normal stress coefficients, Ψ_1 and Ψ_2. Hence suggest an experimental procedure to estimate the value of α.

END OF PAPER