UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 10 June, 2021 12:00 pm to 3:00 pm

PAPER 349

THE LIFE AND DEATH OF GALAXIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **THREE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Dust, is it a nuisance or a blessing? Describe the observable effects of galactic dust on the emitted photons, highlighting the detrimental and beneficial consequences of the interaction of dust and light.

A cloud of dust is composed of identical spherical dust grains, with a single grain having a radius a and a mass m_d. The dust grains emit radiation with emission efficiency Q_ν, which describes what fraction of the black body radiation with intensity $I_\nu = B_\nu$ it produces. An observer on Earth receives flux F_ν from a cloud at a distance D. Assuming that the dust emission is optically thin, i.e. the grains in the front of the cloud do not shield the radiation from the grains in the back, derive a formula that can be used to estimate the total mass of the dust cloud M_d.

Solve the radiative transfer equation $\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$ for a constant, black-body like source function $S_\nu = B_\nu$ and negligible background radiation, i.e. $I_\nu(0) = 0$. Use the solution for intensity I_ν to derive the cloud mass when the emission is not optically thin. Assume that i) the optical depth τ_ν is proportional to the dust column density (number of dust grains per unit area) and emission efficiency Q_ν, and ii) on Earth, the cloud subtends a solid angle Ω.

Show that the general solution reverts to the optically thin case when $\tau_\nu \ll 1$.

Hint: remember the connections between intensity, flux and luminosity!

2

An approximation of the tidal radius (the influence sphere) of a satellite can be obtained by considering the distance from the satellite's centre within which the satellite's average density equals the host's average density within the satellite's orbit.

Describe how the host galaxy potential can be traced if tidal radii of satellites can be measured. Considering globular clusters and dwarf galaxies, which type of satellite system is more likely to be tidally limited?

A more rigorous derivation of the tidal radius requires taking the centrifugal force into account. To derive the tidal radius of a low-mass object orbiting in the gravitational potential of a massive host consider the binary system of two point masses, m and M for the satellite and the host correspondingly. The two objects are separated by distance R_0 and lie at distances x_m and x_M from their centre of mass, so that $x_m + x_M = R_0$. Assume that the satellite is on a circular orbit. By comparing the gravitational forces acting on each of the masses, derive the equation of motion of the separation vector $\mathbf{d} = \mathbf{x_m} - \mathbf{x_M}$, show that it is equivalent to the equation of motion of particle around a combined mass $M + m$, and therefore derive the equation for the orbital frequency of the system Ω.

Consider the equation of motion for a test particle in the vicinity of the satellite, in the reference frame centred on the centre of mass of the system and rotating with angular speed Ω:

$$\ddot{\mathbf{x}} = -\nabla\Phi - 2\boldsymbol{\Omega} \times \dot{\mathbf{x}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$$

This can also be re-written using an effective potential, which combines the contributions of gravitational and the centrifugal forces:

$$\ddot{\mathbf{x}} = -\nabla\Phi_{\rm eff} - 2\boldsymbol{\Omega} \times \dot{\mathbf{x}}$$

Consider the position of a test particle along the x axis near the satellite (with the x axis pointing towards the satellite), i.e. for $y = z = 0$ and find the tidal (or Jacobi) radius r_J, by deriving an equation for the distance at which the derivative of the effective potential is zero, $\frac{\partial\Phi_{\rm eff}}{\partial x} = 0$. In the resulting equation, Taylor expand the $(R_0 - r_J)^{-2}$ term (gravitational potential contribution of the host) in powers of $\frac{r_J}{R_0}$ and keep only the first two terms. In the equation that follows, drop the smallest of the three terms to obtain the equation for r_J.

Hints: Express x_m in terms of m, M and R_0 using the definition of the centre of mass. Take care when taking derivatives of the gravitational potential $\Phi = -\frac{GM}{|x-x'|}$

The above calculation locates the positions of the so-called Lagrange points on either side of the satellite in orbit around the host. Describe the phenomenon that takes place after the stars have left the satellite through apertures near the Lagrange points.

3

Describe the assumptions used to produce the simplest model of the chemical evolution of a galaxy, known as the "closed box" model. As you know, real galaxies do not behave as "closed boxes". Describe the known indirect observational evidence for galactic gas outflows and inflows.

Consider a modification of a "closed box" chemical enrichment model which accounts for both outflows and inflows of gas. Assume constant returned enriched gas mass fraction R and constant yield y_Z (mass of enriched material per generated stellar population of mass $1 - R$). Introduce gas outflows proportional with coefficient λ to the star formation rate $\psi(t)$, and assume that the inflow rate is decaying exponentially with time with a strength A and a characteristic timescale τ. Write down the equations describing the evolution of the galaxy's total mass, gas mass and metal mass, assuming that the accreted gas is primordial, i.e. not enriched with metals. Show that the metallicity Z obeys the following differential equation:

$$\dot{Z} = \frac{y_Z(1-R)\psi(t) + Ae^{-t/\tau}(Z_{\rm prim} - Z)}{M_g(t)},$$

where $M_g(t)$ is the gas mass.

The empirical "star-formation law" can be used as an additional constraint to inform the behaviour of the system described above. Describe the observed law and translating from surface densities to masses, use it to derive the evolution of the galactic gas with time (in the presence of outflows and inflows).

Hint: to simplify the derivation, introduce $\alpha = S(1 - R + \lambda)$, where S describes the connection between the star formation and gas, to arrive at a simple ODE for M_g. Use gas mass at the onset of formation $M_g(0)$ to set the constant of integration.

END OF PAPER