UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 23 June, 2021 12:00 pm to 2:00 pm

PAPER 347

ASTROPHYSICAL BLACK HOLES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider a supermassive black hole of mass $M_{\rm BH}$ accreting material in a steady state. If this material is made up of optically thin ionized hydrogen derive the expression for the Eddington luminosity and explain its physical meaning.

Consider now that the material is made of electron-positron pairs. Derive by how much the Eddington luminosity is changed and explain why this is the case.

A small gas cloud with opacity per unit mass κ is initially at rest at a distance r from this supermassive black hole. Derive the expression for the Eddington luminosity in this case. Suddenly, the black hole's luminosity rapidly increases such that this gas cloud is pushed outwards. From a conservation of energy argument derive the gas cloud velocity at infinity.

(b) Consider an axisymmetric fluid in pure rotation around a central supermassive black hole of mass $M_{\rm BH}$ which dominates the total mass of the system. If the only forces present are gravity, pressure gradients and rotation, write down all components of the momentum equation in cylindrical-polar coordinates and define the effective gravity of this system.

Assuming that radiation pressure is the dominant pressure force and that the fluid is characterized by constant opacity per unit mass κ, relate the flux of radiant energy to the effective gravity in equilibrium.

Consider now a specific geometry of this rotating fluid, assuming it is contained within the surfaces generated by the revolution of two straight lines at angles $\pm\alpha$ to the equatorial plane, where α is not negligible. Sketch the edge-on cross section of this disc and determine the value of effective gravity at the surfaces.

Taking advantage of the geometry of the system, derive that the maximum emitted luminosity by the portion of the disc between two radii R_1 and R_2 (in cylindrical-polar coordinates) is

$$L_{\rm max} = L_{\rm Edd} \sin\alpha \ln(R_2/R_1)\,, \qquad (1)$$

where $L_{\rm Edd}$ is the Eddington luminosity. If R_2 is significantly larger than R_1 explain the physical meaning of $L_{\rm max}$ and of the dependence of $L_{\rm max}$ on α.

2 (a) Consider a dark matter halo well described by a singular isothermal sphere with velocity dispersion σ. A fraction of the total mass of the system $f_{\rm gas}$ is in gas (made of hydrogen atoms) which is in hydrostatic equilibrium. A supermassive black hole of mass $M_{\rm BH}$, which is orbiting this system in equilibrium, is found in the outskirts of the system. Assuming that the dynamical friction exerted by the dark matter distribution alone is causing it to return to the centre, estimate what is the likely gas accretion rate onto this black hole as a function of $M_{\rm BH}$, $f_{\rm gas}$, σ and its distance from the centre r and explain your reasoning.

(b) Assume a steady, spherically symmetric accretion on to a black hole with mass $M_{\rm BH}$. Considering the flow well within the accretion radius $r_{\rm acc} = 2GM_{\rm BH}/c_{\rm s,\infty}^2$, where G is the gravitational constant, and $c_{\rm s,\infty}$ is the gas sound speed at infinity, calculate the optical depth to Thompson scattering τ as a function of accretion rate $\dot{M}$.

At what accretion rate $\dot{M}_{\rm crit}$ is $\tau = 1$ at the Schwarzschild radius? How does $\dot{M}_{\rm crit}$ relate to the Eddington accretion rate $\dot{M}_{\rm Edd}$?

For $\dot{M} > \dot{M}_{\rm crit}$ what happens to the innermost parts of the flow? Calculate the radius of the surface at which $\tau = 1$ as a function of $\dot{M}/\dot{M}_{\rm crit}$.

Furthermore, show that there exists a trapping radius $r_{\rm t}$ within which the inflow speed is higher than the outward photon diffusion speed and calculate its expression as a function of $\dot{M}/\dot{M}_{\rm crit}$. Explain the physical meaning of $r_{\rm t}$.

What is the luminosity, L, that escapes to infinity in the regime $\dot{M} > \dot{M}_{\rm crit}$? How does the radiative efficiency ϵ depend on $\dot{M}/\dot{M}_{\rm Edd}$?

3 (a) Consider a standard, steady, thin accretion disc around a supermassive black hole with mass $M_{\rm BH}$. Adopting cylindrical-polar coordinates write down the expression for the mass conservation equation.

Recalling that radial velocity $u_{\rm R}$ for a thin Keplerian accretion disc is given by

$$u_{\rm R} = -3\frac{\partial/\partial R[\nu\Sigma R^{1/2}]}{\Sigma R^{1/2}}\,, \tag{1}$$

where ν is the kinematic viscosity, Σ is the gas surface density and R is the cylindrical radius, derive an expression for $u_{\rm R}$ in the steady state which depends on ν and R only. [Hint: When calculating the expression for $\nu\Sigma$ think about the appropriate inner boundary condition].

(b) Recalling further that the viscous dissipation $F(R)$ is given by

$$F(R) = \nu\Sigma R^2\left(\frac{d\Omega}{dR}\right)^2, \tag{2}$$

where Ω is the angular velocity, calculate the total luminosity emitted from the disc to be $GM_{\rm BH}\dot{m}/(2R_{\rm ISCO})$ and explain its physical meaning. Here G is the gravitational constant, $\dot{m}$ is the mass accretion rate and $R_{\rm ISCO}$ is the radius of the innermost stable circular orbit.

Calculate the radius inside of which half of the total luminosity is radiated in units of $R_{\rm ISCO}$ and comment on the physical implication of this result.

(c) Once the accretion disc density exceeds the critical value given by $M_{\rm BH}/R^3$ disc self-gravity becomes important. Show that the corresponding critical radius is given by

$$R_{\rm crit} = (3\pi\alpha)^{8/9}(\gamma k_{\rm B}/\mu)^{4/3}\left(\frac{3M_{\rm BH}}{8\pi\sigma_{\rm SB}}\right)^{1/3}(G\dot{m})^{-5/9}\,, \tag{3}$$

where α is the viscosity parameter, γ is the adiabatic index, $k_{\rm B}$ is the Boltzmann constant, μ is the mean molecular weight, and $\sigma_{\rm SB}$ is the Stefan-Boltzmann constant. To derive this expression use the $u_{\rm R}$ from (a) above, keeping only first order terms. Further assume that sound speed can be expressed through the effective temperature which is set by the viscous dissipation from (b), keeping again only first order terms and assuming the black body approximation holds.

END OF PAPER