UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 8 June, 2021 12:00 pm to 3:00 pm

PAPER 346

FORMATION OF GALAXIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Stating any assumptions, derive the linearized Euler equations for a pressureless fluid in comoving coordinates as

$$\frac{d}{dt}(a\mathbf{v}) = -\nabla\Phi,$$

where a is the scalefactor, $\mathbf{v}$ is the peculiar velocity and Φ is the comoving gravitational potential.

Justifying any assumptions, show that at early times

$$\mathbf{v} = -\frac{\nabla\Phi_{\rm i}}{a}\int \frac{D(a)}{a}dt,$$

where $D(a)$ is the linear growth rate. Here, and henceforth, the subscripted roman i refers to an initial or fiducial epoch.

Show that $D(a)$ satisfies

$$\frac{D(a)}{a} = \frac{1}{4\pi G\bar{\rho}_{\rm m,i}}\frac{d}{dt}\left(a^2\frac{dD}{dt}\right),$$

where $\bar{\rho}_{\rm m,i}$ is the mean matter density at the fiducial epoch.

Hence, show that the displacement $\psi(t)$ of any particle is (*the Zeldovich approximation*)

$$\psi(t) = -\frac{D(a)}{4\pi G\bar{\rho}_{\rm m,i}}\nabla\Phi_{\rm i}.$$

The spin of a galaxy arises from the tidal field of all its neighbours. Suppose the material that creates a dark matter halo occupies a Lagrangian volume $V_{\rm L}$ in the early universe. The angular momentum of this material is

$$\mathbf{J} = \int_{V_{\rm L}} d^3\,\mathbf{x}_{\rm i}\bar{\rho}_{\rm m}a^3(a\mathbf{x} - a\bar{\mathbf{x}}) \times \mathbf{v}.$$

where $\bar{\mathbf{x}}$ is the barycentre of the volume and a is the scale-factor. Using the Zeldovich approximation, show that this can be re-written to lowest order as

$$\mathbf{J} = -\bar{\rho}_{\rm m}a^5\dot{b}\int_{V_{\rm L}} d^3\mathbf{x}_{\rm i}\,(\mathbf{x}_{\rm i} - \bar{\mathbf{x}}_{\rm i}) \times \nabla\Phi_{\rm i},$$

for a suitably defined $b(t)$.

Give examples of two instances when $\mathbf{J} = 0$.

[QUESTION CONTINUES ON THE NEXT PAGE]

By Taylor expanding the potential, show that

$$J_i = -a^2 \dot{b} \epsilon_{ijk} T_{jl} I_{lk},$$

where ϵ_{ijk} is the completely antisymmetrical tensor,

$$I_{lk} = \int_V d^3\mathbf{x}_{\rm i}\, (x_{\rm i} - \bar{x}_{\rm i})_l (x_{\rm i} - \bar{x}_{\rm i})_k a^3 \bar{\rho}_{\rm m},$$

and

$$T_{jk} = \nabla_j \nabla_k \Phi_{\rm i}\big|_{\bar{x}_{\rm i}} .$$

Give physical interpretations of the tensors I_{lk} and T_{jk}.

Show that in an Einstein-de Sitter Universe, the modulus of the angular momentum $J \propto t$.

Why is this an underestimate of the final angular momentum of a dark halo in a simulation?

2 The mass function of galaxies is well described by a *Schechter funtion*

$$\phi(M) = \phi_\star \, (M/M_\star)^\alpha \exp(-M/M_\star),$$

where $M_\star$ and $\phi_\star$ are constants and $\alpha \approx -1$. Provide a log-log plot of the function and comment on its behaviour at low and high masses.

Carefully justifying any assumptions, derive *the tensor virial theorem* of a self-gravitating system in the form

$$2K_{ij} + W_{ij} = 0,$$

where the kinetic energy K_{ij} and potential energy W_{ij} tensors are

$$\begin{aligned} K_{ij} &= \frac{1}{2}\int \rho \langle v_i v_j \rangle dV, \\ W_{ij} &= -\int \rho x_i \frac{\partial \Phi}{\partial x_j} dV, \end{aligned}$$

and angled brackets are averages over the distribution.

By denoting the traces as $K = K_{ii}$ and $W = W_{ii}$, show that the total energy E of the system satifies

$$E = -K = \frac{W}{2}.$$

Now consider a system with initial total mass $M_{\rm I}$, total energy $E_{\rm I}$, mean square velocity of stars $\langle v_{\rm I}^2 \rangle$ and gravitational radius $R_{\rm I}$. Show that

$$E_{\rm I} = -\frac{1}{2} M_{\rm I} \langle v_{\rm I}^2 \rangle = -\frac{G M_{\rm I}^2}{2 R_{\rm I}}.$$

Suppose that systems are accreted with energies totalling $E_{\rm A}$, masses totalling $M_{\rm A}$, and mean square speeds averaging $\langle v_{\rm A}^2 \rangle$. If we define the fractions $\eta = M_{\rm A}/M_{\rm I}$ and $\epsilon = \langle v_{\rm A}^2 \rangle / \langle v_{\rm I}^2 \rangle$, show that the final energy of the system is

$$E_{\rm F} = \frac{1}{2} M_{\rm I} \langle v_{\rm I}^2 \rangle (1 + \epsilon \eta),$$

explaining carefully any assumptions made.

Show that the ratio of final to initial mean square speeds is

$$\frac{\langle v_{\rm F}^2 \rangle}{\langle v_I^2 \rangle} = \frac{1 + \eta \epsilon}{1 + \eta}.$$

Show further that the ratio of final to initial gravitational radii is

$$\frac{R_{\rm F}}{R_{\rm I}} = \frac{(1 + \eta)^2}{1 + \eta \epsilon}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

If the total mass of the system increases by a factor of 2 by many minor mergers, show that the density is reduced by a factor of 32.

Observationally, galaxies at a redshift of 2 are smaller in size than present day ellipticals by a factor of three to five, yet their stellar mass densities are an order of magnitude higher. Suggest an explanation of these results using virial arguments and the shape of the Schechter mass function.

3 A simple model of a dark matter halo is *the isothermal sphere* with density and potential

$$\rho(r) = \frac{v_0^2}{4\pi G r^2}, \qquad \phi(r) = v_0^2 \log(r/r_0),$$

where r is spherical polar radius and v_0 and r_0 are constants. Show that the rotation curve is completely flat and that the isotropic velocity dispersion of the dark matter particles is $\sigma = v_0/\sqrt{2}$.

The *Chandrasekhar dynamical friction formula* for a subhalo of mass M moving with velocity $\mathbf{v}$ through a dark matter halo with density ρ is

$$\frac{d\mathbf{v}}{dt} = -\frac{4\pi G^2 M \rho \log\Lambda}{v^3}\left(\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}\exp(-X^2)\right)\mathbf{v},$$

where erf is the error function, $X = v/(\sqrt{2}\sigma)$ and Λ is the Coulomb logarithm. Find the limiting behaviour of the dynamical friction force at low and high speed v, and provide a physical explanation of your results.

Consider a subhalo moving on a circular orbit through a dark halo modelled as an isothermal sphere. Show that the radius of the subhalo's orbit changes with time like

$$r\frac{dr}{dt} = -0.428 \log\Lambda \frac{GM}{v_0}.$$

(Hint: You may assume that $\mathrm{erf}(1) - 2/(e\sqrt{\pi}) = 0.428$*.)*

Hence, show that a subhalo on an orbit of radius r_i sinks to the centre under the effects of dynamical friction on a timescale

$$t_\mathrm{df} = \frac{1.17}{\log\Lambda}\frac{r_\mathrm{i}^2 v_0}{GM}.$$

If, instead, the subhalo is moving on an eccentric orbit, let us define the circularity as

$$\eta = \frac{L}{L_\mathrm{circ}(E)},$$

where E is the energy and L the angular momentum of the orbit, whilst L_circ is the angular momentum of a circular orbit with energy E. Show that, for the isothermal sphere,

$$L_\mathrm{circ}(E) = v_0 r_0 \exp\left[\frac{E - v_0^2/2}{v_0^2}\right].$$

Hence, show that, at fixed position,

$$\frac{d\eta}{dt} = \eta\left[\frac{1}{L}\frac{dL}{dt} - \frac{1}{v_0^2}\frac{dE}{dt}\right].$$

[QUESTION CONTINUES ON THE NEXT PAGE]

If the eccentricity of the subhalo's orbit is e, show that it evolves under the effects of dynamical friction as:

$$\frac{de}{dt} = \frac{\eta}{v}\frac{de}{d\eta}\left[1 - \frac{v^2}{v_0^2}\right]\frac{dv}{dt}.$$

Assuming that η is a monotonically decreasing function of eccentricity e, show that dynamical friction causes the subhalo's orbit to become more circular at pericentre.

How does dynamical friction change the subhalo's orbit at apocentre?

Estimate roughly the overall effect of dynamical friction on the eccentricity?

4 (a) Define the two-point correlation function $\xi(r)$ for a continuous density perturbation field $\delta(\mathbf{x})$, which may be assumed homogeneous and isotropic.

Show that the Fourier transform of the two-point correlation function is the power spectrum

$$P(k) = \int d^3\mathbf{r}\xi(r)\exp(-i\mathbf{k}\cdot\mathbf{r}),$$

where $k = |\mathbf{k}|$.

Show that the variance of the density field is related to the power spectrum by

$$\sigma^2 = \frac{1}{2\pi^2}\int dk\, k^2 P(k).$$

(Hint: You may assume standard theorems in Fourier analysis.)

Using a top-hat window function of volume V, we smooth the density field. Show that the mass variance is related to the correlation function by

$$\sigma^2(M) = \frac{1}{V^2}\int\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \xi(|\mathbf{x}_1 - \mathbf{x}_2|).$$

Express the mass variance in terms of the two-point correlation function and the size R of the top-hat filter.

If the two-point correlation function of galaxies is a power-law

$$\xi(r) = \left(\frac{r}{r_0}\right)^{\gamma},$$

with $r_0 = 5h^{-1}$ Mpc and $\gamma = -1.8$, find an expression for the cosmological parameter σ_8. Here, h is the Hubble constant in units of 100 $\mathrm{kms^{-1}Mpc^{-1}}$.

What assumption underlies this calculation?

(b) Derive the Silk damping scale, or the lengthscale over which a photon can diffuse in time t, as

$$\lambda_{\mathrm{d}} = \left(\frac{ct}{3\sigma_{\mathrm{T}} n_{\mathrm{e}}}\right)^{1/2},$$

where σ_{T} is the Thomson scattering cross-section, n_{e} is the electron number density and c is the velocity of light.

Show that the Silk damping scale in comoving units at the epoch of recombination in an Einstein-de Sitter Universe is

$$\lambda_{\mathrm{d}}^{\mathrm{com}} \approx \frac{6}{5}\left(\frac{ct_{\mathrm{rec}}}{3n_{\mathrm{rec}}\sigma_{\mathrm{T}}}\right)^{1/2}(1 + z_{\mathrm{rec}}),$$

where t_{rec} and z_{rec} are the time and redshift at recombination, whilst n_{rec} is the corresponding electron number density. *(Hint: As recombination takes place during the matter-dominated era, you may ignore the different behavior of the scale-factor during the radiation-dominated era).*

[QUESTION CONTINUES ON THE NEXT PAGE]

Explain the consequences of Silk damping for the growth of perturbations in an Einstein-de Sitter Universe with (i) baryons and radiation, and (ii) baryons, radiation and cold dark matter.

END OF PAPER