UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 10 June, 2021 12:00 pm to 3:00 pm

PAPER 345

FLUID DYNAMICS OF THE ENVIRONMENT

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt ***ALL*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A two-dimensional disturbance is created in an inviscid incompressible Boussinesq fluid with a stable background density stratification $\hat{\rho}(z)$ characterised by no diffusion of mass and constant buoyancy frequency

$$N = \sqrt{-\frac{g}{\rho_0}\frac{d\hat{\rho}}{dz}},$$

where ρ_0 is a reference density and g gravity. The disturbance is described by the velocity field $\mathbf{u}(\mathbf{x},t) = \nabla \wedge (\psi \hat{\mathbf{y}})$, $\psi = \psi(x,z,t)$ and buoyancy perturbation field $b(\mathbf{x},t)$ in a Cartesian coordinate system $\mathbf{x} = (x,y,z)$. Here, t is time, $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the unit Cartesian vectors and $\hat{\mathbf{z}}$ is oriented vertically upwards.

(a) Starting from the equations for conservation of mass and momentum, derive the equations governing a disturbance of arbitrary amplitude in terms of the streamfunction ψ and buoyancy perturbation b. Under what condition(s) can this system be linearised? Show that for a linear disturbance the streamfunction satisfies

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + N^2\frac{\partial^2\psi}{\partial x^2} = 0.$$

(b) Suppose the form of the disturbance is given by $\psi = \epsilon\tilde{\psi}\,e^{i\phi}$ and $b = \epsilon\tilde{b}\,e^{i\phi}$ for $\epsilon > 0$ with constant $\tilde{\psi}, \tilde{b} \in \mathbb{C}$, and $\tilde{\psi}, \tilde{b} = O(1)$. Here, $\phi = \mathbf{k}\cdot\mathbf{x} - \omega t$ with constant $\mathbf{k} = (k, 0, m)$ and $0 < \omega < N$. Without linearising, derive the equation for $\tilde{\psi}$ governing this disturbance and the relationship between $\mathbf{k}$ and ω necessary for a non-trivial solution. Discuss the behaviour of this solution and the role played by ϵ.

(c) Now suppose the form of the disturbance is given by $\psi = \epsilon\tilde{\psi}(\hat{x},\hat{z},\hat{t})\,e^{i\phi}$, where $\hat{x} = \gamma x$, $\hat{z} = \gamma z$ and $\hat{t} = \gamma t$ with $\epsilon, \gamma \ll 1$. Determine the equation governing the evolution of $\tilde{\psi}$ at $O(\epsilon\gamma)$. Consider a disturbance for which initially $\mathbf{k}\cdot\nabla\tilde{\psi} = 0$. At what speed and in what direction does the structure of $\tilde{\psi}$ propagate? Give a physical interpretation of this behaviour.

2

During the winter, a deep lake reaches a uniform temperature $T_0 = 4°\text{C}$ where the density of water is maximum. An accident (in calm conditions) at the centre of the lake releases a volume V of contaminated water at temperature $T_1 = 8°\text{C}$. As the toxic contaminant does not significantly affect the density of the water, we can take the equation of state as $\rho = \rho_0[1 - \beta(T - T_0)^2]$, where $\beta << 1$ is the quadratic thermal expansion coefficient. To leading order, the volume of the contaminated water is conserved, but it cools as it spreads across the lake as an axisymmetric shallow water flow. The evolution of the excess heat content per unit mass of the contaminated layer $H = C_p(T - T_0)$ may be modelled by the heat flux $F_H = KC_p(T - T_0)$ per unit area, where C_p is the constant heat capacity and K is a constant heat transfer coefficient. The role of surface tension and viscosity may be ignored and the lake can be assumed to remain at temperature T_0.

(a) State the conditions required for this flow to be treated as shallow water. Can the flow be considered Boussinesq? (You must justify your answer and comment on the role of the free surface.)

(b) Assuming the flow is axisymmetric, what are the shallow water equations for depth, velocity and non-dimensional temperature excess $\theta = (T - T_0)/(T_1 - T_0)$ governing this flow? Identify the shallow water wave speed c and determine the characteristics λ for the flow. Determine also the equations governing u, θ and c along each of the characteristics.

(c) Why is a front condition necessary to determine the spread of the contaminated water? Use a suitable front condition to derive an integral model for the front location and excess temperature of the current (you may assume that the current depth and excess temperature are the same at all radial locations within the current). What is the maximum area the contaminated water will cover?

3

Consider a well insulated cuboidal room with a vent of area A_f located in the floor ($z = 0$) and a vent of area A_c located in the ceiling ($z = H$). Both vents are connected directly to the outside environment where the density ρ_0 is uniform and there is no wind. The room has a floor area $A_r \gg H^2$. A localised heat source providing buoyancy flux $F_1 > 0$ is located on the floor near the centre of the room. A high-Reynolds-number flow develops above this heat source, ultimately leading to a steady state where there is an interface at $z = h_1$. Above this interface the air density is approximately uniform and equal to ρ_1.

(a) Describe 'Batchelor entrainment' as it may apply to the fluid motion developing above the heat source. State the Boussinesq approximation and discuss any limitations to applying this to the flow. Using the Boussinesq approximation and assuming the heat flux arises from a point source, state a suitable set of equations for describing the flow above the heat source for $z \leqslant h_1$. Give expressions for the volume flux $Q_1(z)$, momentum flux $M_1(z)$ and reduced gravity $g_1'(z)$ within this flow.

(b) Sketch the vertical pressure profile (far from the openings) both inside and outside the room. Establish the conditions necessary for the steady height of the interface to be $h_1 = H/2$.

(c) Suppose a second localised heat source with buoyancy flux $F_2 < F_1$ is introduced to the room and positioned on the floor far from the first heat source. The vents are reconfigured so that $A_c = A_f$ and the original interface remains at $h_1 = H/2$. This second source causes a second interface to form at $z = h_2 < h_1$ with an approximately uniform density ρ_2 in the region $h_2 \leqslant z < h_1$. Sketch key aspects of the steady flow that develops and the pressure fields both inside the room and outside. By assuming the buoyancy associated with F_2 does not change $Q_1(z)$ determined in (a), find h_2/h_1 in terms of F_1 and F_2. Determine $\hat{g}_1'$ and $\hat{g}_2'$, the reduced gravities relative to ambient air of the layers $h_2 \leqslant z < h_1$ and $h_1 \leqslant z < H$, respectively. Derive an explicit expression for the area of the vents required to achieve these conditions (you need not substitute h_1, h_2, $\hat{g}_1'$ or $\hat{g}_2'$ into this expression).

END OF PAPER