UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 24 June, 2021 12:00 pm to 3:00 pm

PAPER 341

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **THREE** questions from questions 1-5 **AND** answer question 6.*
*There are **SIX** questions in total.*
Question 6 carries twice the marks of each of questions 1-5.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, and set

$$\mathbf{g}(\mathbf{y}) = \frac{\partial \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}} \mathbf{f}(\mathbf{y}).$$

We consider the two-step method

$$\mathbf{y}_{n+2} - \frac{8}{11} h\mathbf{f}(\mathbf{y}_{n+2}) + \frac{2}{11} h^2 \mathbf{g}(\mathbf{y}_{n+2}) = \frac{16}{11}\mathbf{y}_{n+1} - \frac{5}{11}\mathbf{y}_n - \frac{2}{11} h\mathbf{f}(\mathbf{y}_n),$$

where $\mathbf{y}_n \approx \mathbf{y}(nh)$.

a. Determine the order of the method.

b. Is the method A-stable?

2

The ODE system $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, is solved by a three-stage Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\frac{3}{5} - \frac{\sqrt{6}}{10} & \frac{3}{25} + \frac{\sqrt{6}}{75} & \frac{1}{5} + \frac{\sqrt{6}}{120} & \frac{7}{25} - \frac{73\sqrt{6}}{600} \\
\frac{3}{5} + \frac{\sqrt{6}}{10} & \frac{3}{25} - \frac{\sqrt{6}}{75} & \frac{7}{25} + \frac{73\sqrt{6}}{600} & \frac{1}{5} - \frac{\sqrt{6}}{120} \\
\hline
 & \frac{1}{9} & \frac{4}{9} + \frac{\sqrt{6}}{36} & \frac{4}{9} - \frac{\sqrt{6}}{36}
\end{array}$$

a. Determine its order, motivating carefully your answer.

b. Is the method algebraically stable?

3

The linear Schrödinger equation

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + V(x)u, \qquad -1 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

is given with periodic boundary conditions and an initial condition for $t = 0$. The potential V is real.

a. Prove that the Euclidean norm $\|u(\,\cdot\,, t)\|$ is constant in t.

b. Suppose that the second space derivative is discretised with second-order central differences, resulting in the semidiscretized scheme

$$\mathbf{u}' = \mathrm{i}(A - V)\mathbf{u},$$

where V is an $(2M) \times (2M)$ diagonal matrix and $V_{k,k} = V((k - M)/M)$, $k = 1, \ldots, 2M$. Write the matrix A explicitly and prove that $\|\mathbf{u}(t)\|$ is constant in t.

c. Propose an efficient implementation of Strang's splitting for the semidiscretized scheme and describe a method for its solution in $O(M \log_2 M)$ operations.

4

The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x},$$

where $\alpha \in \mathbb{R}$, is solved by the semidiscretized scheme

$$u_m' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\alpha}{2\Delta x}(u_{m+1} - u_{m-1}).$$

a. The equation is given for $-1 \leqslant x \leqslant 1$ with zero boundary conditions. Is the scheme stable?

b. The equation is given as a Cauchy problem, i.e. for $x \in \mathbb{R}$. Is the scheme stable?

c. The semidiscretized scheme is solved using the forward Euler method. Write down the scheme. Assuming a Cauchy problem, what is the range of Δt (which may depend on Δx) that ensures stability?

5

We consider the two-point boundary-value problem $\mathcal{L}u = f$, where

$$\mathcal{L}[u] = -(1+x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}u}{\mathrm{d}x} + x^2u$$

given in $[0,1]$ with zero Dirichlet boundary conditions.

a. Specifying in which function space should the solution reside, prove that $\mathcal{L}$ is positive definite and determinate the underlying variational problem.

b. The problem is solved with the Ritz method, using hat functions. Letting $f(x) \equiv 1$, determine the underlying system of algebraic linear equations for the coefficients. (No need to evaluate the integrals.)

6

Write an essay on the concept of stability in the numerical solution of time-dependent differential equations.

You should explain the rationale behind the definitions, quote relevant theorems, describe techniques to determine stability, their scope and limitations, and accompany your presentation with concrete examples.

END OF PAPER