UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2021 12:00 pm to 2:00 pm

## PAPER 336

## PERTURBATION METHODS

***Before you begin please read these instructions carefully***

*Candidates have TWO HOURS to complete the written examination.*

*Attempt no more than* ***TWO*** *questions.*
*There are* ***THREE*** *questions in total.*
*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** The function $g(\theta;\lambda)$ is defined by

$$g(\theta;\lambda) = \int_C \frac{1}{(z-z_0)(z^2-1)^{\frac{1}{2}}} \exp\left(i\lambda\left((z^2-1)^{\frac{1}{2}}\theta - z\right)\right)\, dz\ ,$$

where $\theta$, $z_0$ and $\lambda$ are real and positive, and the contour $C$ goes from $-\infty$ to $\infty$ passing above the three singularities of the integrand. Take the branch cuts for $(z^2-1)^{\frac{1}{2}}$ to be lines drawn towards $\Im(z) = -\infty$ from the points $z = \pm 1$.

(a) Obtain the leading-order asymptotic behaviour of $g(\theta;\lambda)$ as $\lambda \to \infty$ on the assumption that $0 < \theta < 1$ and $z_0 > 1$; be careful to discuss all cases. Determine the limiting value of $g(\theta;\lambda)$ as $\theta \to 1$.

(b) Comment, with brief justification, on the value of $g(\theta;\lambda)$ when $\theta > 1$.

(c) If $\theta = 1$, outline, without performing a detailed calculation, how you would obtain a leading-order approximation of $g(\theta;\lambda)$ as $\lambda \to \infty$.

**2**

(a) For $0 \leqslant x \leqslant 1$, the function $y(x;\varepsilon)$ satisfies the differential equation

$$\varepsilon^2 \frac{d^2y}{dx^2} = y - \left|x - \tfrac{1}{2}\right| ,$$

where $0 < \varepsilon \ll 1$, together with the boundary conditions

$$y(0;\varepsilon) = y(1;\varepsilon) = \tfrac{1}{2} .$$

Find an asymptotic solution of the composite form

$$y(x;\varepsilon) \sim y_0(x) + \varepsilon y_1(\xi) ,$$

where $\xi = \left(x - \frac{1}{2}\right)/\varepsilon$. What is the order of smoothness of this composite asymptotic expansion?

(b) For $0 \leqslant x \leqslant 1$, the function $z(x;\varepsilon)$ satisfies the differential equation

$$\sigma\varepsilon^2 \frac{d^2z}{dx^2} + 4x\ln^2 x \, \frac{dz}{dx} - 4\sigma\, z = 0 ,$$

where $0 < \varepsilon \ll 1$ and $\sigma$ is a constant, together with the boundary conditions

$$z(0;\varepsilon) = z(1;\varepsilon) = 1 .$$

(i) If $\sigma = -1$, determine the leading-order solution for $z$ in inner and outer regions that are to be identified.

*Hint: Depending on your approach, you may find it helpful to know that the general solution to*

$$\mu \frac{d^2Y}{d\xi^2} + 4\xi \ln^2 \xi \, \frac{dY}{d\xi} = 0 ,$$

*where $\mu$ is a constant, is*

$$Y = c + d \int_0^{\xi} \exp\left(-\frac{\xi^2}{\mu}\left(2\ln^2\xi - 2\ln\xi + 1\right)\right) d\xi ,$$

*where $c$ and $d$ are constants.*

(ii) If $\sigma = 1$, outline how the form of the leading-order solution changes, and sketch the solution.

**3** For $t \geqslant 0$, the function $\theta(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2\theta}{\partial t^2} - k^2\frac{\partial^2\theta}{\partial x^2} - \theta = \varepsilon^2\theta^3\,,$$

where $k$ is a positive constant and $\varepsilon \ll 1$ is a small positive constant. At $t = 0$, the function $\theta$ satisfies the initial conditions

$$\theta(x,0) = \cos x\,, \qquad \frac{\partial\theta}{\partial t}(x,0) = 0\,.$$

Assume that for $k \geqslant k_c(\varepsilon)$, this system admits time periodic solutions with a real frequency and phase. By seeking an asymptotic solution of the form

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots\,,$$

find $k_c(\varepsilon)$ correct to, and including, terms that are $\mathcal{O}(\varepsilon^2)$.

*Hint: before scaling time, it may be helpful to note the form of the linear solution when* $0 \leqslant k - k_c(0) \ll 1$.

*Comment: the following trigonometric relation may prove useful*

$$4\cos^3\xi = 3\cos\xi + \cos 3\xi\,.$$

## END OF PAPER