UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2021 12:00 pm to 3:00 pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For an unbounded, inviscid, stratified fluid in a rotating reference frame, consider small amplitude departures from a basic state with velocity $\mathbf{u} = \Lambda x \hat{\boldsymbol{\jmath}}$ and buoyancy $b = N^2 z$, where $\hat{\boldsymbol{\jmath}}$ is the unit vector in the y direction and Λ and N are constant. You may assume that the perturbations are independent of y and that the Coriolis parameter, f, is constant.

(i) Show that the frequency, ω, of small amplitude sinusoidal perturbations satisfies

$$\omega^2 = f\widetilde{f}\frac{m^2}{k^2+m^2} + \widetilde{g}\frac{k^2}{k^2+m^2},$$

where f is the Coriolis parameter, k and m are the wavenumbers in the x and z directions, and $\widetilde{f}$ and $\widetilde{g}$ are functions that should be determined.

Discuss how the dispersion relation varies as Λ changes and obtain a condition for exponentially growing perturbations. Write down a necessary condition for instability, expressed in terms of the Ertel potential vorticity of the basic state.

(ii) Now, consider a basic state with buoyancy of the form $B = M^2 x + N^2 z$, where M and N are constant. The velocity of the basic state is a superposition of the velocity in the part above ($\Lambda x \hat{\boldsymbol{\jmath}}$) and a vertical shear in thermal wind balance with the horizontal buoyancy gradient. Derive the dispersion relation for small amplitude sinusoidal perturbations that are independent of y and with velocity aligned with surfaces of constant buoyancy (such that the perturbation buoyancy vanishes).

Obtain a condition for exponentially growing perturbations and express this in term of the Ertel potential vorticity of the basic state. Discuss the role of the horizontal buoyancy gradient (M^2) on the condition for instability. Compare this result with part (i) and discuss the connection in terms of the component of the basic state vorticity perpendicular to constant buoyancy surfaces.

2

A thin, flat sheet of ice floats on the surface of the ocean ($z = 0$). At a sufficiently large height, the wind in the atmosphere is in geostrophic balance and has velocity $\mathbf{U}_a$. At a sufficiently large depth, the ocean is in geostrophic balance and has velocity $\mathbf{U}_o$. Let the constant kinematic viscosity of the atmosphere and ocean be ν_a and ν_o, respectively, and assume that the density of the atmosphere, ρ_a, and the density of the ocean, ρ_o, are constant. Find the velocity of the ice when the system is in steady state. You may neglect the thickness of the ice for the purpose of applying boundary conditions. Clearly state any assumptions that you make. Discuss the solution in the limit when $\nu_a \to 0$.

The atmospheric Ekman transport is defined as the vertically-integrated steady-state wind,

$$\overline{\mathbf{u}}_a = \int_0^\infty (\mathbf{u}_a(z) - \mathbf{U}_a)\, dz. \tag{1}$$

In conditions when $\mathbf{U}_o$ is aligned with $\mathbf{U}_a$, find the orientation of the atmospheric Ekman transport and the orientation of the stress exerted by the wind on the ice, relative to the geostrophic flow.

3 Quasi-geostrophic flow on a β-plane with Coriolis parameter $f_0 + \beta y$ and constant buoyancy frequency N evolves according to the quasi-geostrophic potential vorticity equation

$$\frac{D_g}{Dt}\left\{\psi_{xx} + \psi_{yy} + \frac{f_0^2}{N^2}\psi_{zz}\right\} + \beta\psi_x = 0$$

where ψ is the quasi-geostrophic stream function and D_g/Dt denotes the rate of change following the geostrophic flow. The leading-order approximation to the vertical velocity is given by

$$w = -\frac{D_g}{Dt}\left\{\frac{f_0\psi_z}{N^2}\right\}.$$

Consider flow in an ocean, unbounded in x and y, with, in its resting state, a free surface at $z = 0$ and a rigid lower boundary at $z = -H$. In the disturbed state the free surface is $z = \eta(x, y, t)$ and the velocity is non-zero. Assume that the pressure is constant above the free surface, and that the free surface displacement η is small enough that linearisation applies.

(i) Show that the appropriate boundary condition at $z = 0$ is:

$$\frac{D_g\psi_z}{Dt} + \frac{N^2}{g}\frac{D_g\psi}{Dt} = 0.$$

(ii) Show that the quasi-geostrophic potential vorticity equation, linearised about the resting state, has a solution for ψ of the form:

$$\psi(x, y, z, t) = \phi(x, y, t)P(z)$$

provided that $\phi(x, y, t)$ satisfies a certain partial differential equation and $P(z)$ satisfies the eigenvalue relation:

$$\frac{d^2P}{dz^2} = -\frac{N^2}{c^2}P. \tag{1}$$

For given c clearly state the partial differential equation satisfied by ϕ.

(iii) What boundary conditions apply to $P(z)$ at $z = -H$ and $z = 0$? Hence obtain an equation for c (which you will not be able to solve analytically). (You may assume c to be positive.)

(iv) In the case where N^2H/g is small obtain leading-order expressions for the possible values of c, i.e. c_n, $c_n = 0, 1, 2, \ldots$, with $c_0 > c_1 > c_2 \ldots$ *[Hints: You may find it useful to consider the variable NH/c rather than c and you may also find a graphical approach useful to locate solutions.]* The eigenfunction $P_n(z)$ satisfies (1) with $c = c_n$. Give the leading-order expressions for $P_0(z)$ and $P_1(z)$.

[QUESTION CONTINUES ON THE NEXT PAGE]

Now consider the effect of a forcing term $F(x, y, z)H(t)$, where $F(x, y, z) = \sum_{n=0}^{\infty} F_n(x, y)P_n(z)$, on the right-hand side of the quasi-geostrophic potential vorticity equation. $H(t)$ is the Heaviside step function. For each z the function $F(x, y, z)$ is non-zero only in some finite region $\mathcal{R}$ (which includes the origin) of the (x, y)-plane, with length scale L_h.

The corresponding solution of the quasi-geostrophic potential vorticity equation may be written as

$$\psi(x, y, z, t) = \sum_{n=0}^{\infty} \phi_n(x, y, t)P_n(z).$$

(v) Derive the partial differential equation satisfied by each ϕ_n and then the simplified 'long-wave' form of the equation that applies if $L_h \gg c_n/f$ for each $n = 0, 1, 2, \ldots$.

(vi) Give a qualitative description of the solution of the simplified equation, for a forcing of the type specified above. What will be the vertical structure of ψ observed at two points X_1 and X_2, which are respectively a large distance L_X to the west and to the east of the forcing region $\mathcal{R}$, for times t such that $c_0 t > L_X > c_1 t$?

(vii) Suppose in particular that $F(x, y, z) = G(x, y) \neq 0$ for $-D < z < 0$ and $F(x, y, z) = 0$ for $-H < z < -D$. What is the steady-state solution for ψ? *[Hint: Consider the steady-state form of the quasi-geostrophic potential vorticity equation without writing ψ as a series.]* Give a brief description of the evolution towards this steady state.

4 Consider the Boussinesq equations for small-amplitude disturbances to a basic state with constant velocity $U > 0$ in the x direction and with constant buoyancy frequency N:

$$\mathbf{u}'_t + U\mathbf{u}'_x = -{\rho_0}^{-1}\nabla p' + b'\hat{\mathbf{z}} \tag{1}$$

$$\nabla \cdot \mathbf{u}' = 0 \tag{2}$$

$$b'_t + Ub'_x + w'N^2 = 0. \tag{3}$$

Quantities vary only in x and z. $\hat{\mathbf{z}}$ is the unit vector in the upward vertical direction and $\mathbf{u}' = (u', w')$.

(i) Consider disturbances with sinusoidal variation in x, z and t of the form $e^{i(kx+mz-\omega t)}$. Derive the dispersion relation giving the frequency ω in terms of the wavenumber (k, m). (Note that the dispersion relation has two branches.) Identify clearly the conditions on k and m that correspond to upward or downward group propagation.

(ii) Now consider flow above a sinusoidal boundary defined by $z = h_0 \cos kx$ where h_0 is small. Assume that the effect of the boundary is to provide a forcing at frequency $\omega = 0$. What is the linearised boundary condition on w' at $z = 0$? Under what conditions is there upward propagation of waves away from the boundary? Sketch the possible configurations for the lines of constant phase in this case.

(iii) The force on the x-averaged flow due to the waves is given by $-\partial(\overline{u'w'})/\partial z$. For conditions where there is wave propagation, calculate the momentum flux $\overline{u'w'}$ in terms of h_0, k and U. If the waves dissipate in some layer a large distance above the boundary what will be the corresponding force (integrated over the dissipation layer) exerted on the x-averaged flow?

(iv) Explain the overall momentum balance in the flow.

(v) By considering $\partial(\overline{b'\zeta'})/\partial t$, where $\zeta' = u'_z - w'_x$ derive an x-averaged wave-activity conservation relation in which $-\overline{u'w'}$ appears as a vertical flux. Give the corresponding expression for the wave activity density.

(vi) Use the wave-activity conservation relation to explain the relation between the growth or decay of waves and the force exerted on the x-averaged flow. Assuming that growing waves imply a force with the same sign as you have deduced in (iii) above, what do you deduce about the sign of the wave activity? How would your answer to (iii) and your answer here change if $U < 0$? (Detailed calculation is not required.)

END OF PAPER