UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2021 12:00 pm to 2:00 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A fluid of viscosity μ_1 and density ρ_1 is injected upwards into a two-dimensional porous medium of porosity ϕ and uniform permeability k that is aligned vertically and that is saturated in a second fluid of viscosity μ_2 and density ρ_2. The initial interface between the two fluids is planar, and the porous medium is thin, so that at all times the flow is principally two-dimensional and in the plane of the porous medium. If the fluid is injected with constant velocity U, show that the interface is unstable to perturbations with growth rate

$$\sigma = \frac{\alpha U}{\phi}\left\{M + \frac{(\rho_2 - \rho_1)kg}{(\mu_1 + \mu_2)U}\right\} \tag{1}$$

where $M = (\mu_2 - \mu_1)/(\mu_2 + \mu_1)$ is the mobility ratio, α is the horizontal wavenumber of the perturbations and g is the gravitational acceleration.

If the two fluids are immiscible, with interfacial surface tension γ, find the most unstable wavenumber as a function of the mobility ratio M, the capillary number $Ca = \mu_2 U/\gamma$ and a ratio between the characteristic buoyancy velocity and the imposed velocity U. Derive an expression for the most unstable wavenumber and plot the growth curve, $\sigma(\alpha)$, for the cases $\rho_2 = \rho_1$, $\rho_2 > \rho_1$ and $\rho_2 < \rho_1$. Finally, find a condition on the density, ρ_2, for which the interface is neutrally stable ($\sigma = 0$) and comment on your result, in particular in the limit as $U \to 0$.

2

(a) Water of temperature T_m occupying $z > 0$ is brought into contact with a cold substrate of initial temperature $T_s < T_m$ and thermal conductivity k_s occupying $z < 0$ to form a layer of ice of conductivity k occupying $0 < z < h(t)$, where T_m is the freezing temperature of water. Determine $h(t)$ up to a factor λ, an implicit function of $\mathcal{S} = L/c_p(T_m - T_s)$, where L is the latent heat of fusion and c_p is the specific heat capacity, which may be assumed to be independent of material. Determine also the contact temperature $T_0 = T(z = 0)$ as a function of λ. Simplify your results in the asymptotic limits (i) $k_s \gg k$ and (ii) $k_s \ll k$, and explain your findings physically. In particular, in case (ii), find an explicit expression for $h(t)$ in terms of the parameters of the system.

(b) A layer of ice sits below a layer of salt solution containing a total mass of salt S per unit area, all contained in a narrow cell. Above the salt solution is an essentially infinite layer of fresh water. The system is pulled at constant speed V past a heat exchanger that imposes a fixed temperature T_c at its location while the cell sits in an environment of temperature $T_\infty > T_m$, where T_m is the freezing temperature of pure water. Assume that the heat flux from the environment to the cell is directly proportional to the temperature difference between the environment and the interior of the cell, with constant of proportionality $\mathcal{F}$.

Show that the system can achieve a steady state in the frame of reference of the heat exchanger provided $T_c < T_m - mSV/\rho D$, where m is the slope of the liquidus (assumed linear), ρ is the density of the salt solution (assumed undefended of salt concentration), and D is the diffusivity of salt in solution. Determine expressions for the temperature field in each region and explain how the steady height of the ice front above the heat exchanger could be calculated from these expressions.

Determine the conditions under which constitutional supercooling occurs in the liquid region. In particular, sketch the critical solidification rate V as a function of S in the two limits $\mathcal{F}/k \ll V^2/\kappa^2$ and $\mathcal{F}/k \gg V^2/\kappa^2$, where k is thermal conductivity and κ is thermal diffusivity.

3

A two-dimensional ice sheet of thickness $H(x,t)$, where x is horizontal distance and t is time, erodes the rock over which it flows to form a lubricating layer of *till* of thickness $h(x,t)$. You may assume that the ice and till both flow as Newtonian fluids, with dynamic viscosities μ and $\lambda\mu$ respectively, $\lambda \ll 1$, and the same densities ρ. You may also assume from the outset that $h \ll H$. The rate of erosion of the rock is given by $E\tau_0$, where τ_0 is the viscous shear stress exerted on the rock.

Starting from the equations for thin-film flow, give a careful derivation of the leading-order equations, in the limit $h/H \ll 1$,

$$\frac{\partial H}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left[\left(1+\frac{3h}{\lambda H}\right)H^3\frac{\partial H}{\partial x}\right], \tag{1}$$

$$\frac{\partial h}{\partial t} = \frac{\rho g}{2\lambda\mu}\frac{\partial}{\partial x}\left[Hh^2\frac{\partial H}{\partial x}\right] - \rho g E H\frac{\partial H}{\partial x}. \tag{2}$$

Consider the steady state that arises if ice is supplied with volume flux q_0 per unit cross-stream width at $x=0$ and is removed at rate q_0 at $x=L$ (for example by calving at a continental margin). Use a scaling analysis to show that $h \sim (\lambda\mu E L)^{1/2}$, independent of q_0, and that the ice is essentially unlubricated if

$$\mathcal{E} \equiv \frac{\mu E^2 L\rho g}{\lambda^2 q_0} \ll 1.$$

Solve the steady equations for a well-lubricated ice sheet ($\mathcal{E} \gg 1$) to show that $F = H/H_0$ satisfies

$$3F^4 - 2F^6 = 1-\xi,$$

where $H_0 = H(0)$ and $\xi = x/L$, and determine the value of H_0 in terms of the parameters of the system. Determine the thickness $h(x)$ of the till in terms of F, and draw sketches of $H(x)$ and $h(x)$.

END OF PAPER