UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 23 June. 2021 12:00 pm to 2:00 pm

PAPER 331

HYDRODYNAMIC STABILITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Hydrodynamic Stability

You are given that rotating Rayleigh-Benard convection in an infinite layer of Boussinesq fluid is governed by the following dimensionless equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \lambda \hat{\mathbf{z}} \times \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \sigma Ra\, \theta\, \hat{\mathbf{z}} + \sigma \nabla^2 \mathbf{u},$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \nabla^2 \theta$$

with the velocity $\mathbf{u}$ satisfying stress-free boundary conditions at $z = 0, 1$ and the temperature field $\theta = -1$ on $z = 1$ and $\theta = 0$ on $z = 0$. Ra is the Rayleigh number, σ is the Prandtl number and λ is a non-dimensional measure of the rotation rate.

(a) Show that $\mathbf{u}_0 = 0$ and $\theta_0 = -z$ give a basic conductive state.

(b) Write down the linearized Boussinesq equations for small perturbations of $(\mathbf{u}_0, \theta_0)$ together with the relevant boundary conditions.

(c) By taking $\hat{\mathbf{z}} \cdot \nabla \times$ and $\hat{\mathbf{z}} \cdot \nabla \times \nabla \times$ of the linearized momentum equation, derive the equations

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2 \right) \omega - \lambda \frac{\partial W}{\partial z} = 0,$$

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2 \right) \nabla^2 W + \lambda \frac{\partial \omega}{\partial z} = \sigma\, Ra\, \nabla_H^2 \theta'$$

where $W := \mathbf{u} \cdot \hat{\mathbf{z}}$, $\omega := \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$ and θ' is the temperature perturbation to θ_0. Here $\nabla_H^2 := \nabla^2 - \partial^2/\partial z^2$ and you can use the identity $\hat{\mathbf{z}} \cdot \nabla \times \nabla \times \mathbf{A} = \hat{\mathbf{z}} \cdot \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 (\mathbf{A} \cdot \hat{\mathbf{z}})$.

(d) Now consider normal modes of the form $[W, \omega, \theta'] = [\hat{W}(z), \hat{\omega}(z), \hat{\theta}(z)] e^{\mu t + ikx}$. By eliminating $\hat{\omega}$ and $\hat{\theta}$, deduce that

$$\left[(\mu - \hat{D}^2)(\mu - \sigma \hat{D}^2)^2 \hat{D}^2 + \lambda^2 D^2 (\mu - \hat{D}^2) + \sigma Ra k^2 (\mu - \sigma \hat{D}^2) \right] \hat{W}(z) = 0$$

where $D := \partial/\partial z$ and $\hat{D}^2 := D^2 - k^2$.

(e) Assuming that linear instability first appears for $\mu = 0$, find the threshold Rayleigh number as a function of λ, σ and k. Is the effect of rotation stabilizing or destabilizing?

(f) Find an expression for the optimal k when $\lambda/\sigma \gg 1$.

(g) Slightly above the stability point Ra_c, the evolution of $A(t)$, the amplitude of the convection, is given by the equation

$$\frac{dA}{dt} = (Ra - Ra_c) A - A^3.$$

Illustrate on a graph of $A(t)$ versus t how the solutions behave as a function of the initial condition $A(0)$ (be careful to indicate the stability of any steady solutions).

2 Hydrodynamic Stability

(i) Consider the stability of the 2-dimensional *inviscid* parallel shear flow

$$\mathbf{u} = U(y)\mathbf{e}_x \qquad -1 \leqslant y \leqslant 1.$$

(a) If $\psi = \phi(y)e^{ik(x-ct)}$ is the stream function of the perturbation velocity, derive Rayleigh's stability equation.

(b) State the boundary conditions on ϕ for rigid walls, located at $y = \pm 1$ and show that the boundary condition for a free, constant pressure surface is $(U - c)\phi' - U'\phi = 0$.

(c) For the case $U(y) = y$, show that for discrete modes

$$c^2 = \frac{(k \tanh k - 1)(k - \tanh k)}{k^2 \tanh k}$$

for a free surface condition at $y = \pm 1$ and hence deduce that the flow is unstable for $k < k_c$ where the condition defining k_c should be given.

(ii) Now consider the system

$$\frac{\partial^2 u}{\partial t^2} + f(u) = \frac{1}{R}\frac{\partial^2 u}{\partial z^2} \qquad \text{for} \quad 0 < z < \pi$$

and $u = 0$ at $z = 0, \pi$ where $f(0) = 0$, $f'(0) < 0$, $f''(0) = 0$ and $f'''(0) > 0$.

(a) Show that the solution $u = 0$ is linearly stable if $R \leqslant R_c = -1/f'(0)$.

(b) Consider the weakly nonlinear problem when $\varepsilon^2 := R - R_c \ll 1$. Assuming $u = \varepsilon A(T)\sin z + \varepsilon^2 u_2(z, T) + \varepsilon^3 u_3(z, T) + \ldots$, show that $u_2 = 0$ and that

$$\frac{d^2 A}{dT^2} = \frac{1}{f'(0)^2}A - \frac{1}{8}f'''(0)A^3$$

where $T := \varepsilon t$ and

$$\int_0^\pi u_j(z, T)\sin z\, dz = 0 \qquad \text{for} \quad j = 2, 3, \ldots$$

to ensure A is uniquely defined.

3 Hydrodynamic Stability

Consider the 2D linear system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} -\varepsilon & -(1+\beta) \\ (1-\beta) & -\varepsilon \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\beta < 1$ and ε are positive constants.

(a) Find the condition on β and ε for the origin to be linearly stable.

(b) Show that L is non-normal for $\beta \neq 0$ but that growth in the energy $E(t) := x(t)^2 + y(t)^2$ is only possible if $\beta > \varepsilon$.

(c) When $\varepsilon = 0$, show that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A(t) \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} := \begin{bmatrix} \cos \Gamma t & -(1+\beta)/\Gamma \sin \Gamma t \\ \Gamma/(1+\beta) \sin \Gamma t & \cos \Gamma t \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

where $\Gamma := \sqrt{1-\beta^2}$.

(d) Again for $\varepsilon = 0$, show that the maximum energy growth, $E(t)/E(0)$, after a time t is the largest solution, λ, of the equation

$$(\lambda - 1)^2 = \frac{4\beta^2}{1-\beta^2}\,\lambda \sin^2 \Gamma t.$$

Hence deduce *when* the growth is maximized, find the optimal initial conditions for this and compute the maximum growth possible.

(e) Relate your results from (d) to the solution trajectories around the origin.

END OF PAPER