UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 8 June, 2021 12:00 pm to 3:00 pm

PAPER 329

SLOW VISCOUS FLOW

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **THREE** questions in total.*
The questions carry equal weight.

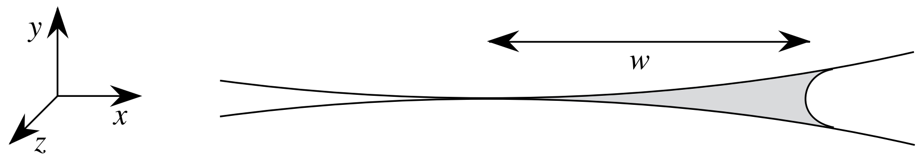
STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Slow Viscous Flow

Two infinite rigid cylinders of radius a are parallel and touching along the z-axis. A small amount of viscous fluid occupies the cusp-shaped region on one side of the cylinders, and the wetting properties of the fluid are such that the meniscus is tangent to the cylinders at the contact points, as shown in the diagram. The width of fluid $w(z,t)$ satisfies $w \ll a$ and varies slowly in the axial direction z.



Making appropriate geometrical approximations, show that the cross-sectional area of fluid is proportional to w^3 and that the curvature of the semicircular meniscus is proportional to w^{-2}. Find the constants of proportionality.

Gravity is negligible and the fluid is drawn (in both directions) along the cusp between the cylinders by the variation in the capillary pressure. By integrating the flux over the cross-section, derive the equation

$$\frac{\partial w^3}{\partial t} = \frac{\gamma}{7\mu a}\frac{\partial}{\partial z}\left(w^4 \frac{\partial w}{\partial z}\right)$$

for the evolution of $w(z,t)$.

Obtain a similarity solution for the spread of a small fixed volume V of fluid placed in the cusp at $z=0$ and $t=0$. In particular, show that the location of one tip of the flow is at

$$z_N(t) = \left(\frac{8Va}{\pi}\right)^{1/4}\left(\frac{8\gamma t}{21\mu a}\right)^{3/8}.$$

Suppose that gravity is no longer negligible and that the cylinders are now placed, still touching, with their axes vertical in a large bath of fluid whose free surface is at $z=0$. Find $w(z)$ for large z after the flow has come to rest.

2 Slow Viscous Flow

A small spherical bubble of radius a is immersed in fluid of viscosity μ. The fluid contains dissolved surfactant with volumetric concentration $\Gamma(\mathbf{x})$, which has a weak uniform far-field gradient $\mathbf{G} = (\boldsymbol{\nabla}\Gamma)^{\infty}$. The surfactant is also adsorbed onto the surface of the bubble at a rate given by

$$-k(C - b\Gamma),$$

where C is the surface concentration of surfactant sitting on the interface, Γ is the adjacent bulk concentration, and k and b are constants. Explain with a sketch why you would expect the bubble to move. [*Hint*: The motion is called chemophoresis.]

The diffusivity of surfactant in the bulk fluid is D. Write down the governing transport equation for Γ and give a physical interpretation of the boundary condition

$$D\mathbf{n}\boldsymbol{\cdot}\boldsymbol{\nabla}\Gamma = -k(C - b\Gamma), \qquad (r = a),$$

where $\mathbf{n}$ is the unit normal out of the bubble.

Neglecting advection of the fluid, what does the symmetry of the problem suggest is the form of Γ. Assuming that $kab \ll D$ and that C has a similar magnitude to $b\Gamma$ (both on average and in variation), show that $\Gamma = \frac{3}{2}a\mathbf{G}\boldsymbol{\cdot}\mathbf{n} + \Gamma_0$ on $r = a$, where Γ_0 is a constant.

The surface-tension coefficient on the interface is given by $\gamma(C) = \gamma_0 - \gamma_1 C'$, where γ_0 and γ_1 are positive constants, $C' = C - C_0$ and $C_0 = b\Gamma_0$. Write down the general stress boundary condition for a fluid–fluid interface with surface tension γ and curvature κ. Assume that $C' = A\mathbf{G}\boldsymbol{\cdot}\mathbf{n}$ for some constant A and find the resultant fluid stress $\boldsymbol{\sigma}\boldsymbol{\cdot}\mathbf{n}$ at $r = a$.

Verify that the net force exerted by the bubble on the fluid is zero, and explain briefly why this should be the case. Explain why the flow induced by the Marangoni stress can be determined from Papkovich–Neuber potentials of the form

$$\chi = \beta a^3\, \mathbf{G}\boldsymbol{\cdot}\boldsymbol{\nabla}\,\frac{1}{r}\,, \qquad \boldsymbol{\Phi} = \mathbf{0}\,,$$

where β is a constant. On $r = a$ the stress corresponding to χ is

$$\boldsymbol{\sigma}\boldsymbol{\cdot}\mathbf{n} = \frac{3\beta}{a}\big\{\mathbf{G} - 3(\mathbf{G}\boldsymbol{\cdot}\mathbf{n})\mathbf{n}\,\big\}.$$

Find β and show that the fluid velocity $\mathbf{u}$ is consistent with uniform translation of the bubble at a velocity $\mathbf{U} = \gamma_1 A\mathbf{G}/3\mu$.

In a steady state the usual transport equation for the surface concentration C simplifies to

$$\boldsymbol{\nabla}_s\boldsymbol{\cdot}(C\mathbf{u}_s) = D_s\nabla_s^2 C - k(C - b\Gamma)\,,$$

where $\mathbf{u}_s = \mathbf{u} - \mathbf{U}$. Assume that $|C'| \ll C_0$ and simplify the equation further. Hence determine the constant A in the earlier assumption $C' = A\mathbf{G}\boldsymbol{\cdot}\mathbf{n}$.

[*You may use the results* $\nabla_s^2\mathbf{n} = -2\mathbf{n}/a^2$ *and* $\boldsymbol{\nabla}_s\boldsymbol{\cdot}(\mathbf{I} - \mathbf{nn}) = -2\mathbf{n}/a$.]

For each of the parameters k, C_0 and D_s, explain the physical mechanism by which $\mathbf{U}$ is increased or decreased by increasing that parameter.

3 Slow Viscous Flow

(a) A rigid cylinder of radius a and length L is surrounded by viscous fluid bounded by a rigid plane $z = 0$. The axis of the cylinder is given by $x = 0$, $0 \leqslant y \leqslant L$ and $z = a(1 + \frac{1}{2}\epsilon)$, where $\epsilon \ll 1$.

Use lubrication theory to show that the components A_{xx} and A_{yy} of the resistance matrix associated with translations of the cylinder in the x and y directions, respectively, are approximated to leading order by

$$A_{xx} = 4\pi\epsilon^{-1/2}\mu L \quad \text{and} \quad A_{yy} = 2\pi\epsilon^{-1/2}\mu L\,.$$

$\Big[$ *You may assume that if* $I_n \equiv \displaystyle\int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^n}$ *then* $I_1 = \pi$, $I_2 = \dfrac{\pi}{2}$ *and* $I_3 = \dfrac{3\pi}{8}$. $\Big]$

(b) Now consider a rigid torus of cross-sectional radius a and centreline given by $x^2 + y^2 = R^2, z = a(1 + \frac{1}{2}\epsilon)$, where $R \gg a$. Use the answers to part (a) to calculate the leading-order approximations to the components A_{xx} and D_{zz} of the resistance matrix for the torus.

(c) Now suppose that the fluid is confined between rigid planes at $z = 0$ and $z = a(2 + \epsilon)$ and consider translation of the torus from part (b) with velocity $(-U, 0, 0)$. Assume, to begin with, that the flux leaking through the narrow gaps near $r = R$ is negligible.

Working in the frame of the torus, find the Hele-Shaw flows and the pressure distributions in $r > R$ and in $r < R$. [For this calculation, you should ignore the details for $r - R = O(a)$ and approximate the second plane to be at $z = 2a$.]

Sketch the pressure distribution in plan view, showing regions of high and low pressure.

Deduce that if $a \ll R \ll a\epsilon^{-1/2}$ then the resistance to translation of the torus is approximately twice the answer to part (b), but if $a\epsilon^{-1/2} \ll R \ll a\epsilon^{-5/2}$ then the resistance is approximately

$$12\pi\mu U R^2/a\ .$$

Use scaling arguments to explain the significance of the condition $R \ll a\epsilon^{-5/2}$.

END OF PAPER