UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 11 June, 2021 12:00 pm to 2:00 pm

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the space of test functions $\mathcal{D}(\mathbf{R})$ and the space of distributions $\mathcal{D}'(\mathbf{R})$, specifying the notion of convergence on each.

Show that a linear form $u : \mathcal{D}(\mathbf{R}) \to \mathbf{C}$ defines an element of $\mathcal{D}'(\mathbf{R})$ if and only if $\langle u, \varphi_m \rangle \to 0$ for each sequence of test functions $\{\varphi_m\}_{m \geqslant 1}$ that converge to zero in $\mathcal{D}(\mathbf{R})$.

For $h \in \mathbf{R}$ and $u \in \mathcal{D}'(\mathbf{R})$ define the translation $\tau_h u$ and the derivative u'. Show that both define elements of $\mathcal{D}'(\mathbf{R})$.

Show that

$$u' = \lim_{h \to 0} \frac{\tau_{-h} u - u}{h} \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

For $-1 < \lambda < 0$ define the locally integrable functions on $\mathbf{R}$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x \leqslant 0. \end{cases} \qquad \log x_+ = \begin{cases} \log x, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Show that for $\varphi \in \mathcal{D}(\mathbf{R})$

$$\left\langle (x_+^\lambda)', \varphi \right\rangle = \int_0^\infty \big[\varphi(x) - \varphi(0)\big] \lambda x^{\lambda - 1} \, \mathrm{d}x.$$

Derive a similar expression for $\langle (\log x_+)', \varphi \rangle$.

Find the order of $(x_+^\lambda)'$, justifying your answer.

2

Let $X \subset \mathbf{R}^n$ be open. What does it mean for a function $\Phi = \Phi(x,\theta)$ to be a *phase function*? Define the space of symbols $\mathrm{Sym}(X, \mathbf{R}^k; N)$ and show that:

(i) If $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ then $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbf{R}^k; N - |\beta|)$.

(ii) If $a_i \in \mathrm{Sym}(X, \mathbf{R}^k; N_i)$ for $i = 1, 2$ then $a_1 a_2 \in \mathrm{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

For Φ a phase function and $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ define

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x,\theta)\, \mathrm{d}\theta$$

in terms of a linear map from $\mathcal{D}(X)$ to $\mathbf{C}$. You may assume $I_\Phi(a) \in \mathcal{D}'(X)$. Define the *singular support* of an element of $\mathcal{D}'(X)$ and show that

$$\operatorname{sing\,supp} I_\Phi(a) \subset \{x \in X : \nabla_\theta \Phi(x,\theta) = 0 \text{ for some } \theta \in \mathbf{R}^k \setminus \{0\}\}.$$

Let $(x,k) \in \mathbf{R}^3 \times \mathbf{R}^3$. Define $u \in \mathcal{D}'(\mathbf{R}^3)$ by the oscillatory integral

$$u(x) = \frac{1}{2\pi} \int \frac{e^{\mathrm{i}k\cdot x}}{1 + |k|^2}\, \mathrm{d}k.$$

Using spherical polars $(k_1, k_2, k_3) = (r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta)$ with volume element $\mathrm{d}k = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$, show that for $x \in \mathbf{R}^3 \setminus \{0\}$ the distribution u can be identified with the function

$$x \mapsto \frac{\pi e^{-|x|}}{|x|}.$$

Comment on this result.

Hint: You may use the fact that for $\lambda > 0$

$$\int_0^\infty \frac{r\sin(\lambda r)}{1 + r^2}\, \mathrm{d}r = \frac{\pi}{2} e^{-\lambda}.$$

3

Define the Schwartz space of functions $\mathcal{S}(\mathbf{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$. Show that the Fourier transform defines a continuous isomorphism $\mathcal{F} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$. Hence show that the Fourier transform extends to a continuous isomorphism on the space of tempered distributions.

(a) For $\omega \in \mathbf{R}$, compute the Fourier transform in $\mathcal{S}'(\mathbf{R})$ of the function

$$x \mapsto \exp\left(\frac{\mathrm{i}}{2}\omega x^2\right).$$

(b) For a real, invertible matrix $A \in \mathrm{GL}(n)$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$ define the pull-back $(A^*\varphi)(x) = \varphi(Ax)$. Using a duality argument, show that this definition extends to $u \in \mathcal{S}'(\mathbf{R}^n)$ via

$$\langle A^*u, \varphi\rangle = \frac{1}{|\det(A)|}\left\langle u, (A^{-1})^*\varphi\right\rangle \quad \forall \varphi \in \mathcal{S}(\mathbf{R}^n).$$

You may assume $A^*u \in \mathcal{S}'(\mathbf{R}^n)$.

(c) Show that for $u \in \mathcal{S}'(\mathbf{R}^n)$

$$(A^*u)\hat{} = \frac{((A^t)^{-1})^*\hat{u}}{|\det(A)|}.$$

(d) Deduce that for a real, symmetric matrix $A \in \mathrm{GL}(n)$

$$\left[\exp\left(\frac{\mathrm{i}}{2}Ax\cdot x\right)\right]\hat{}(\lambda) = \sqrt{\frac{(2\pi)^n}{|\det(A)|}}\exp\left(\frac{\mathrm{i}\pi}{4}\,\mathrm{sgn}(A) - \frac{\mathrm{i}}{2}(A^{-1}\lambda\cdot\lambda)\right),$$

where $\mathrm{sgn}(A) = \sum_{i=1}^n \mathrm{sgn}\,\xi_i$, with $\{\xi_i\}$ being the eigenvalues of A. You may assume elementary results from linear algebra.

END OF PAPER